Modeling with Differential Equations: Introduction to the Issues

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- A differential equation is an equation involving derivatives and functions.

- Assume that we have a complete description of the derivative of a function in the form of an equation that it satisfies.

- Question: What is the function? That is, how can we obtain f from f' ?

- Many physical and biological systems can be modeled with differential equations.

- Often it is relatively easy to measure the amount of something that is present at a given time, and then how the amount changes as the system goes from one state to another.

Example

- The rate of increase of a large rabbit population is proportional to the number of rabbits at that time.

- If we let $y(t)$ be the number of rabbits at time t, then

$$\frac{dy}{dt} = ky,$$

  where k is a constant of proportionality.

- If we let $y(0)$ be the number of rabbits at the beginning of the observation period, then we say that we have a differential equation and an accompanying initial condition.

Solution by Inspection

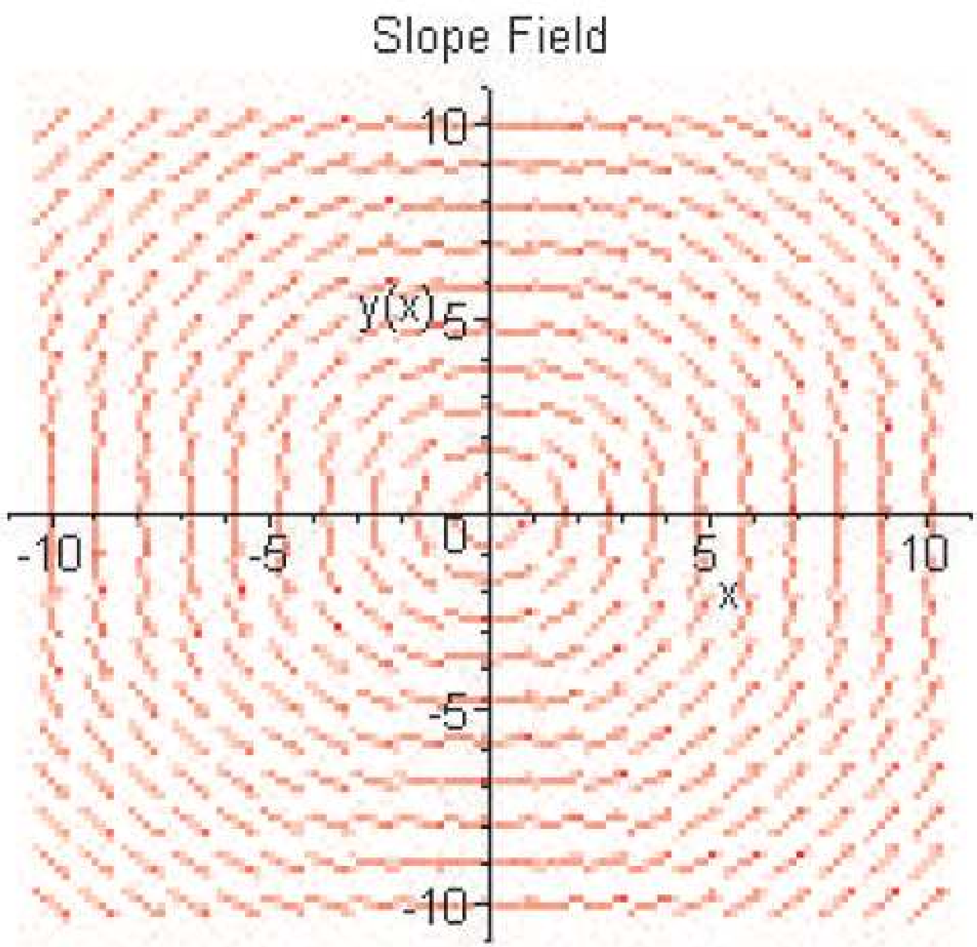
- Let
$$\frac{dy}{dt} = ky.$$
- Is there an elementary function whose derivative is a constant times itself?

Slope Fields

- In the guess-and-check method, we are considering the equation from a formulaic point of view.
- We could also think about the equation from the perspective of slopes.
- Example:

$$\frac{dy}{dx} = -\frac{x}{y}$$

- A plot of the tangent lines is called the *slope field* of the differential equation.

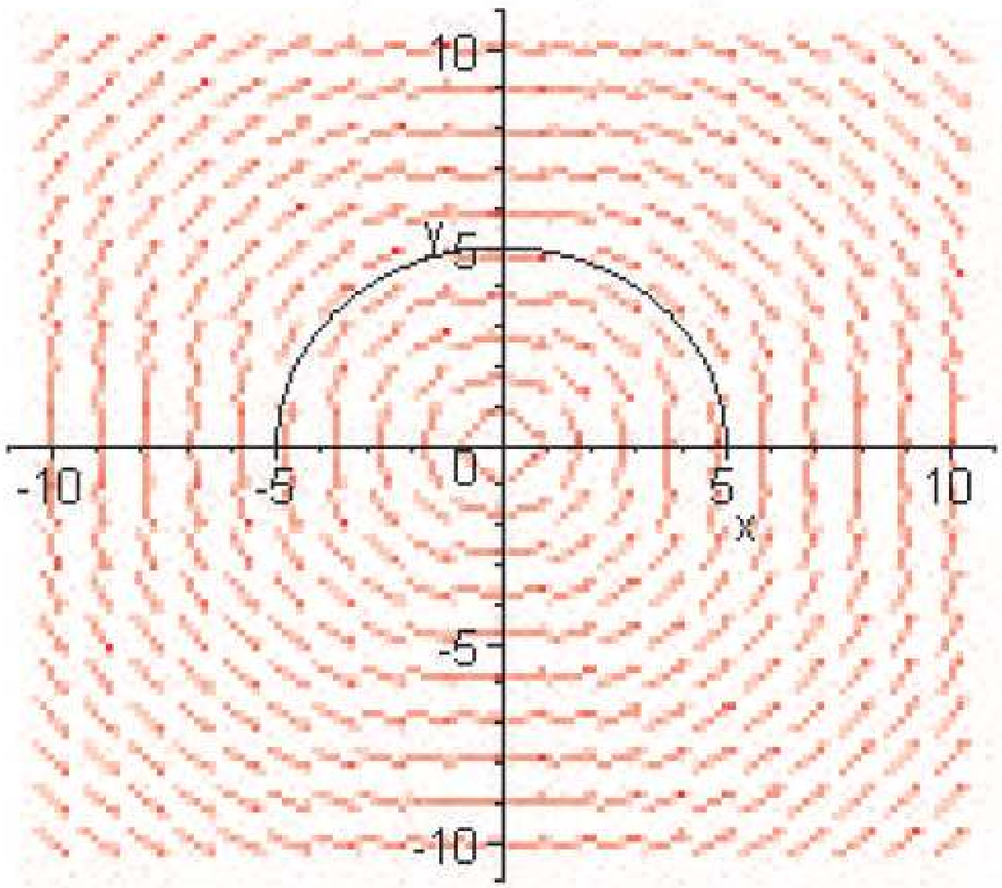


Slope Field
10
y(x) 5
-5
-10
-10
-5
0
5
10
x

- From a starting point, say $(-5, 0)$, it looks like the curve that follows the slopes is a semicircle centered at the origin.

- the slope field suggests that a solution function might be $y = \sqrt{a^2 - x^2}$

Slope Field of y'=-x/y



An Analytical Tool: Separation of Variables

- We considered Initial Value Problems (IVPs).

- They involved differential equations that can be put in the form

$$\frac{dy}{dx} = g(x); y(a) = y_a.$$

- We can solve explicitly for the derivative as a function of x.

- The general solution is found by integrating both sides of the equation and using the initial condition to determine the particular solution.

Example

$$\frac{dy}{dx} = x^2 + x + 1; y(0) = 2$$

$$\frac{dy}{dx} = e^x + \frac{1}{x}; y(1) = e$$

Separable Equations

- They are of the form

$$\frac{dy}{dx} = g(x) \cdot h(y).$$

- A separable equation is one in which we can put all of the y's and dy's (as products) on one side of the equation and all of the x's and dx's (as products) on the other:

$$\frac{dy}{h(y)} = g(x)dx$$

Examples

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$\frac{dy}{dx} = ky$$

Not every simple-looking equation is separable

$$\frac{dy}{dx} = x - y$$

Existence and Uniqueness of Solutions of Initial Value Problems

- In most cases we will meet, a solution to an IVP will exist and will be unique.
- Consider the IVP $y' = F(x, y), y(a) = b$, where $F(x, y)$is continuous in a domain D.
- An important result in the theory of differential equations is Peano's Existence Theorem, which states there is always at least one solution of the IVP, and any such solution is differentiable.

- Even though our techniques will produce a solution, how do we know that it is the only one?

- How do we know that there is not another solution different from the one that we get from applying the method of separation of variables to a separable equation?

Answer

- We consider a differential equation $y' = g(x)h(y)$.

- Assume g and h are continuous in a region D containing the initial point (a, b). Then there is always a solution (Peano), and if g' and h' are continuous, the solution is unique.