Slope Fields and Euler's Method

11/02/2005

- We are going to study the geometric information that we get from a differential equation that gives an explicit formula for the derivative.

- Consider the differential equation

$$\frac{dy}{dx} = F(x, y); y(x_0) = y_0.$$

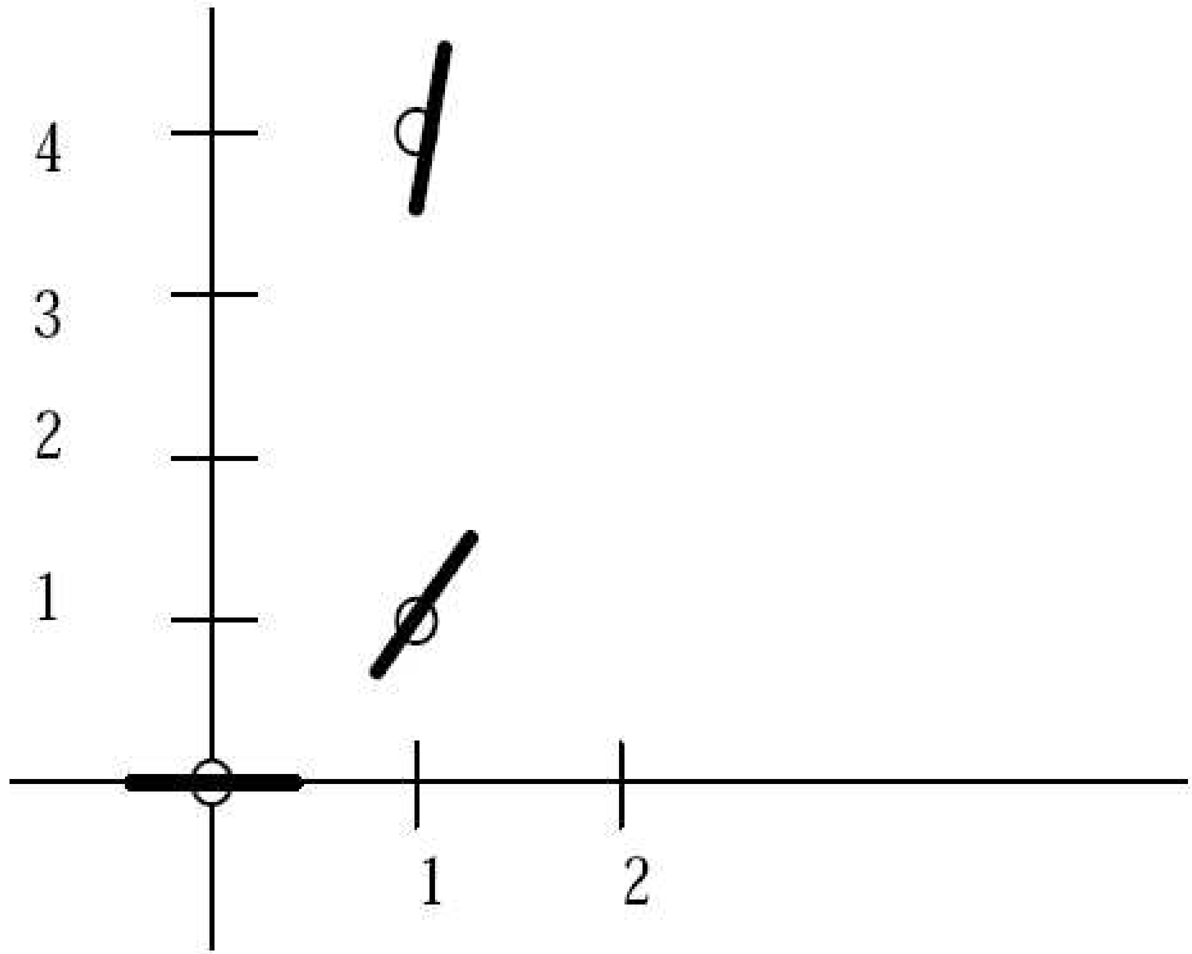
- The equation says that at any point (x, y) in the plane we can compute the slope $\frac{dy}{dx}$ of the tangent line through that point.

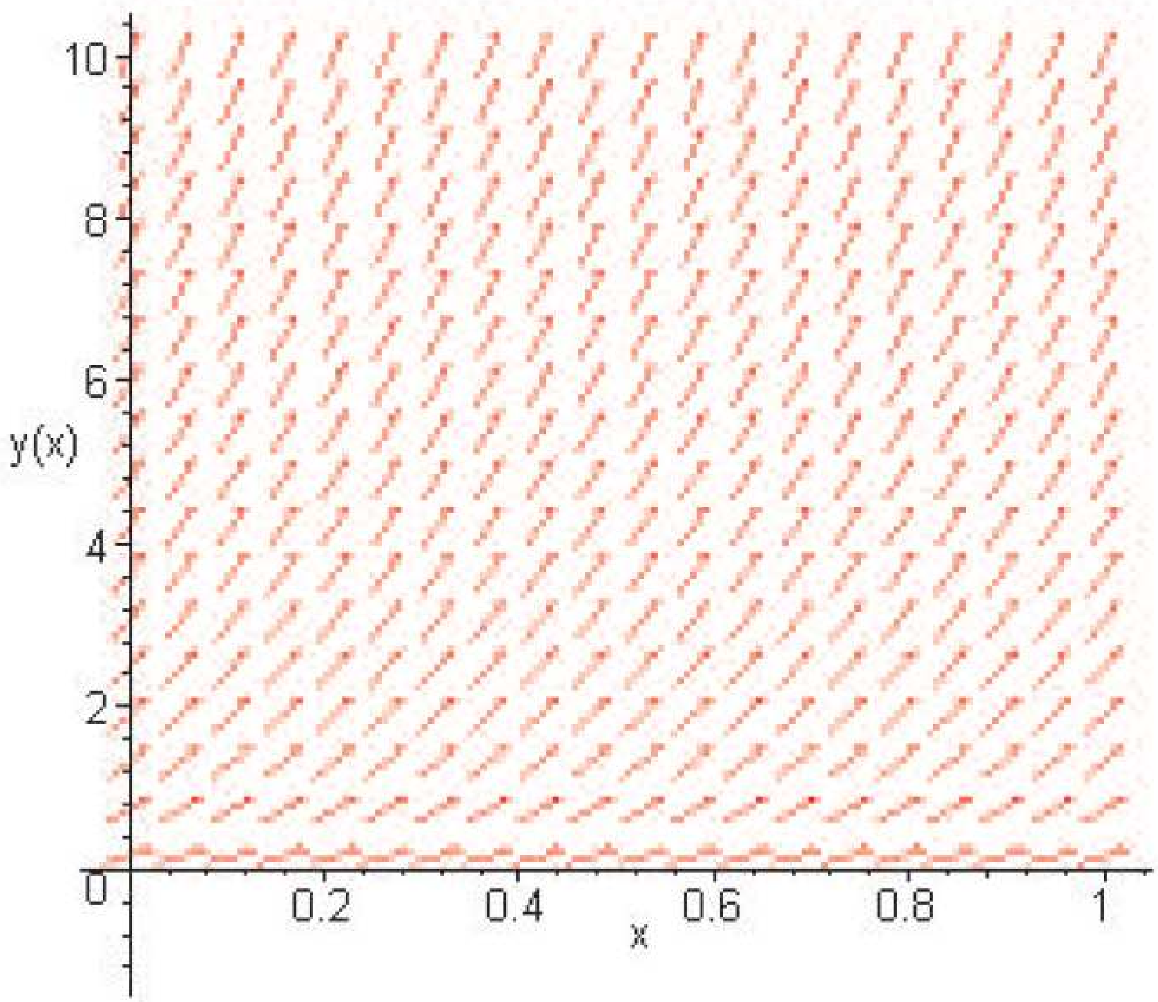
- At each point (x, y) in the plane, we can draw a short straight line whose slope is $F(x, y)$ from the differential equation.

- The resulting two-dimensional plot of tangent lines is called the slope field or direction field of the differential equation.

- Let $F(x, y) = 8\sqrt{y}$

Point (x, y)	Slope $F(x, y)$
$(0, 0)$	0
(1,1)	8
(1,4)	16


4
3
2
1
1
2



10
8
6
4
2
0
y(x)
0.2
0.4
0.6
0.8
1
x

- Can you guess the shape of the solution curve that passes through $(0, 0)$?

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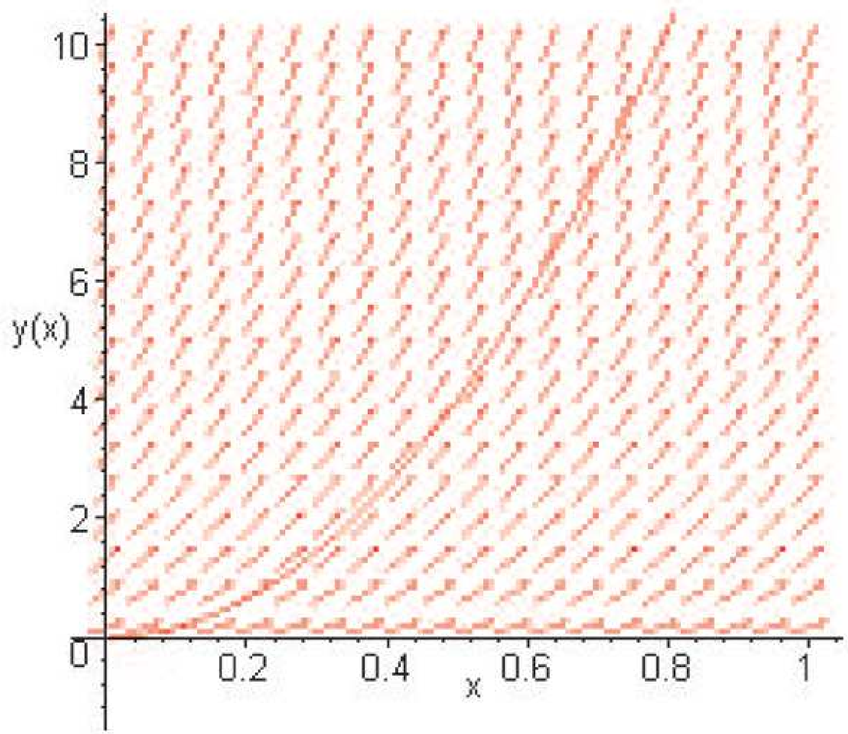

Example

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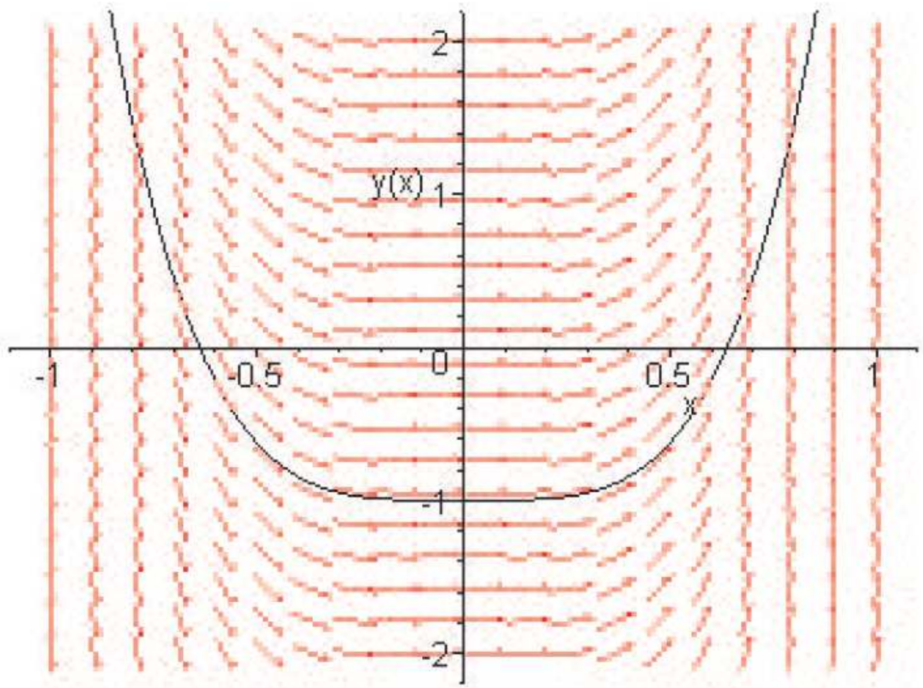


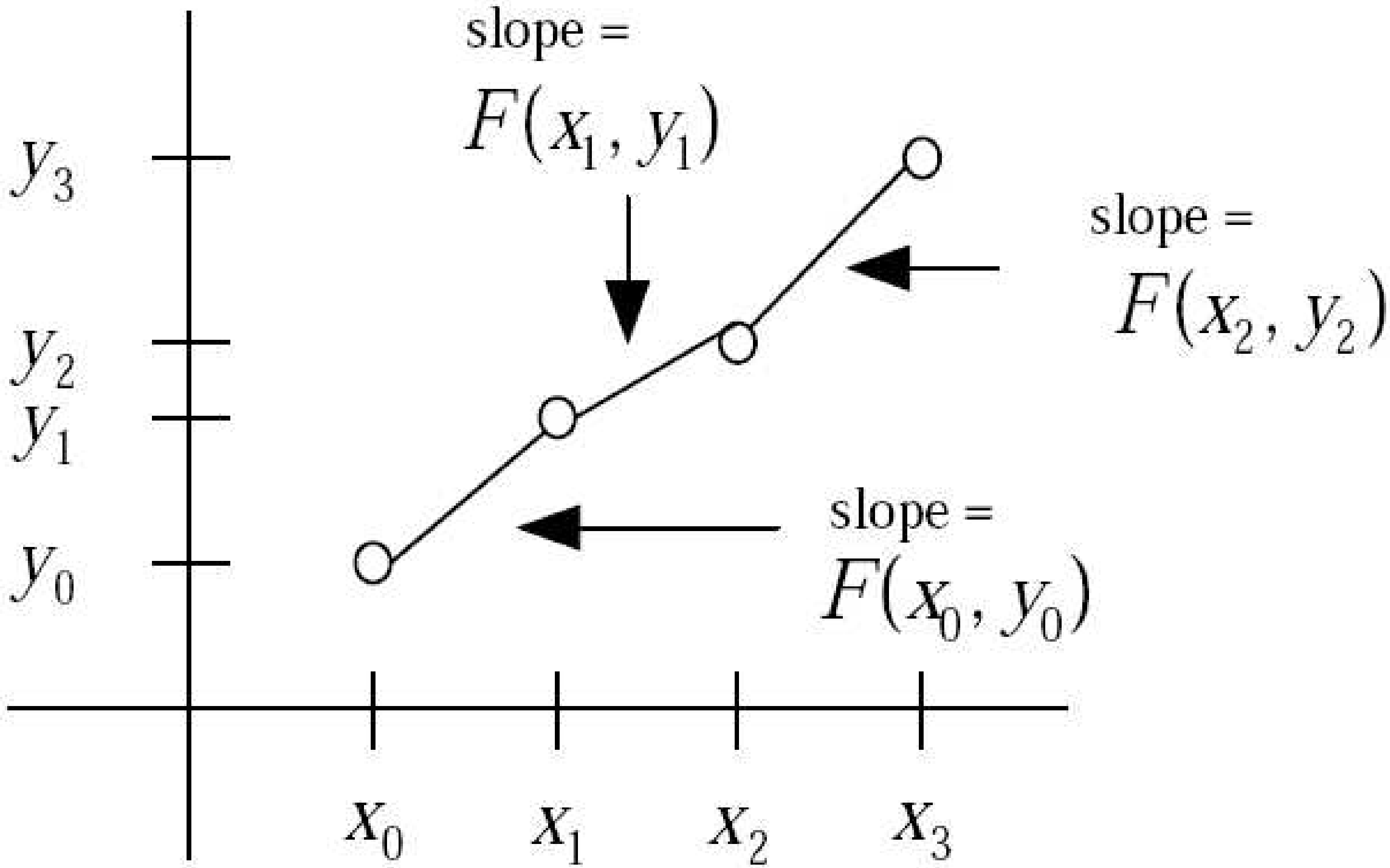
Euler's Method

- Assume that the following IVP is given:

$$\frac{dy}{dx} = F(x, y); P_0 = (x_0, y_0)$$

- The method consists of starting at the initial point $P0 = (x0, y0)$, specifying an increment x, and plotting a sequence of line segments joined end to end.

- The slope of each segment is the value of the derivative at the initial point of the segment.


slope =
$F(x_1, y_1)$
slope =
$F(x_2, y_2)$
slope =
$F(x_0, y_0)$
y_3
y_2
y_1
y_0
x_0
x_1
x_2
x_3

Theorem. *Given the Initial Value Problem*

$$\frac{dy}{dx} = F(x, y); P0 = (x0, y0),$$

and Δx specified, then the endpoints of the line segments that make up the polygonal path in Euler's Method are

$$\begin{aligned} x_{n+1} &= x_n + \Delta x \\ y_{n+1} &= y_n + \Delta x F(x_n, y_n) \end{aligned}$$

Example

- Let $\frac{dy}{dx} = x - y$; $y(0) = 1$.
- On the interval $[0, 1]$ approximate $y(1)$ with two steps of size $1/2$.
- Here $F(x, y) = x - y$ and $\Delta x = 1/2$.

Example

- Let $\frac{dy}{dx} = x - y$; $y(0) = 1$.
- On the interval $[0, 1]$ approximate $y(1)$ with two steps of size $1/2$.
- Here $F(x, y) = x - y$ and $\Delta x = 1/2$.
- Thus, $y_1 = y_0 + \Delta x F(x_0, y_0) = 1 + (1/2)(-1) = 1/2$;
- $y_2 = y_1 + \Delta x F(x_1, y_1) = 1/2 + (1/2)(0) = 1/2$.
- Therefore $y(1) \simeq 1/2$.

Case Study: Population Modeling

- **Objective:** Predict the size of the US population well into the 21st century.

- Translate real-world problems into mathematical models.
- Subject the models to mathematical analysis and prediction.
- Draw conclusions from the models.
- Test the conclusions in the laboratory and compare the results with the original real-world data.
- Revise the model as necessary and repeat the above steps until the model is a reliable predictor of real-world observations.

The Malthus and Verhulst Models

- The **Malthus** model for growth of a population assumes an ideal environment.
- Resources are unlimited, disease is constrained, and individuals are happy.
- The population increases at a rate proportional to the number of individuals present.

- The $Verhulst$ model assumes that the growth rate declines, from a value k when conditions are very favorable, to the value 0 when the population has increased to the maximum value M that the environment can support.

- Takes into account the effects of a limiting environment.

- It is a more realistic model.

- We will use only the populations recorded in the census of 1790 and of 1990.

Year	Population
1790	3.9
1990	250

The Malthus Model: Exponential Growth

- Starting with a population of 3.9 million in 1790, we have the Initial Value Problem

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- Have fun with the applet!

The Verhulst Model: Limited Exponential Growth

- The Verhulst model assumes that the growth rate declines, from a value k when conditions are very favorable, to the value 0 when the population has increased to the maximum value M that the environment can support.

- It replaces the growth constant k by the expression

$$k\frac{M - Q(t)}{M}$$

- This leads to the differential equation

$$\frac{dQ}{dt} = k\frac{M - Q}{M}Q.$$

- The factor $\frac{M-Q}{M}$ that has a value between 0 and 1 is sometimes called the unrealized potential for population growth.

- When Q is small it has a value close to 1, and the growth of the population is essentially exponential.

- As Q approaches its asymptotic limiting value, however, the factor $\frac{M-Q}{M}$ is close to zero, and the population grows ever more slowly.

Objective

- The U.S. population cannot sustain exponential growth indefinitely.
- The Malthus model gives unrealistic projections of the population over the next century.
- We would like to use the Verhulst model instead to make such projections.
- We also need to assume that $Q(0) = 3.9$ million, and $M = 750$ million, the maximum value of the population ($0 \leq Q(t) \leq M$).

HAVE FUN!

- And don't forget to use the applets!