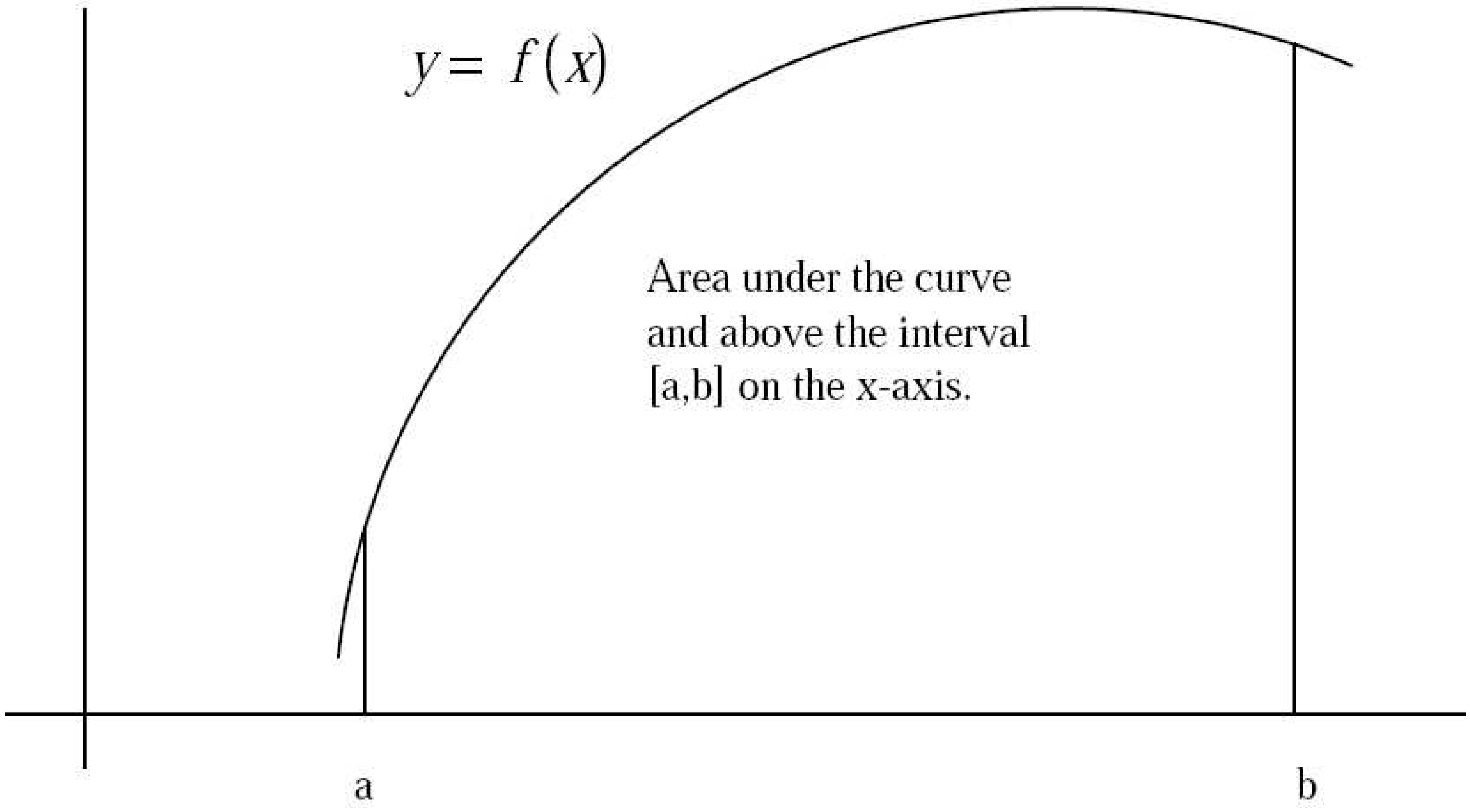
The Definite Integral

11/08/2005

The Area Problem

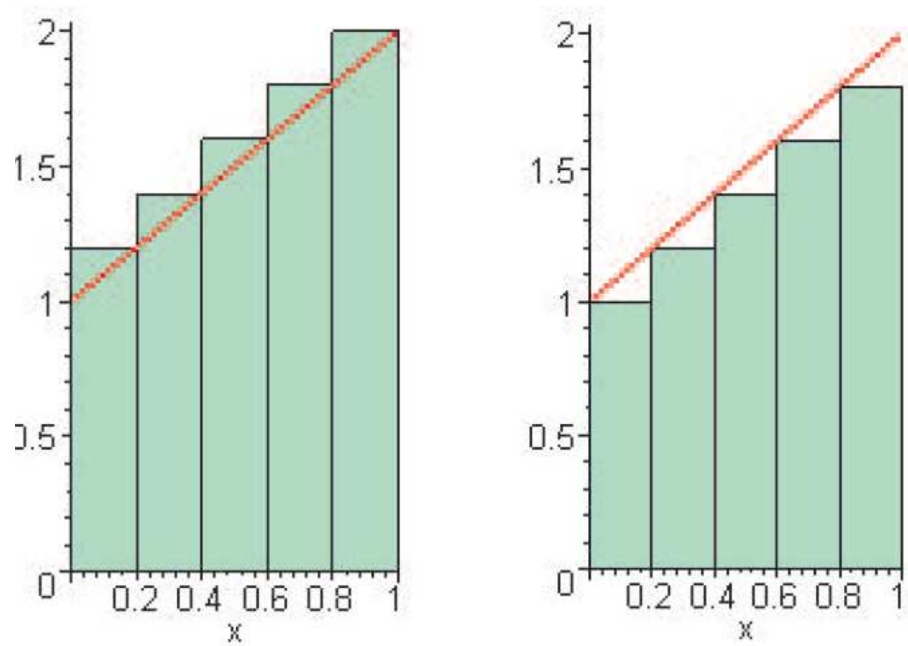


Assumptions about Areas

1. Area is a nonnegative number.
2. The area of a rectangle is its length times its width.
3. Area is additive. That is, if a region is completely divided into a finite number of non-overlapping subregions, then the area of the region is the sum of the areas of the subregions.

Upper and Lower Sums; the Method of Exhaustion

Suppose we want to use rectangles to approximate the area under the graph of $y = x + 1$ on the interval $[0, 1]$.



- We will call the sum of the areas of the rectangles in the left picture an *Upper Riemann Sum*, and the sum of the areas of the rectangles in the right picture a *Lower Riemann Sum*.

- The Upper Sum $= 31/20$ and Lower Sum $= 29/20$.

- The process of increasing the number of rectangles to improve the approximation to the area whose value we seek is reminiscent of the *Greek Method of Exhaustion*.

n	U	L
100	1.505000000	1.495000000
150	1.503333333	1.496666667
200	1.502500000	1.497500000
300	1.501666667	1.498333333
500	1.501000000	1.499000000

General Procedure for finding the Area Under a Curve and Above an Interval

1. Let $y = f(x)$ be given and defined on an interval $[a, b]$. Subdivide the interval $[a, b]$ into n subintervals. Label the endpoints of the subintervals $a = x_0 \leq x_1 \leq x_2 \leq x_3 \cdots \leq x_n = b$. Define $P = \{x_0, x_1, x_3, \ldots, x_n\}$ to be a *partition* of $[a, b]$.

2. Let $\Delta x_i = x_i - x_{i-1}$ be the width of the i^{th} subinterval, $1 \leq i \leq n$.

3. Form the Upper Riemann Sum $U(P, f)$: the height of each rectangle is the *maximum* value M_i of the function on that i^{th} subinterval.

$$U(P, f) = M_1 \Delta x_1 + M_2 \Delta x_2 + M_3 \Delta x_3 + \cdots + M_n \Delta x_n$$

4. Form the Lower Riemann Sum $L(P, f)$: the height of each rectangle is the *minimum* value m_i of the function on that i^{th} subinterval.

$$L(P, f) = m_1 \Delta x_1 + m_2 \Delta x_2 + m_3 \Delta x_3 + \cdots + m_n \Delta x_n$$

5. Take the limit as $n \to \infty$ and the maximum $\Delta x_i \to 0$.

Sigma Notation

If m and n are integers with $m \leq n$, and if f is a function defined on the integers from m to n, then the symbol $\sum_{i=m}^{n} f(i)$, called sigma notation, is defined to be $f(m) + f(m+1) + f(m+2) + \ldots + f(n)$.

Example

1. $\sum_{i=1}^{n} i = 1 + 2 + 3 + 4 \cdots + n$
2. $\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + 4^2 \cdots + n^2$
3. $\sum_{i=1}^{n} 1 = \underbrace{1 + 1 + 1 + 1 \cdots + 1}_{ntimes}$

The Area Problem Revisited

$$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i$$

$$L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i,$$

where M_i and m_i are, respectively, the maximum and minimum values of f on the ith subinterval $[x_{i-1}, x_i]$, $1 \leq i \leq n$.

Riemann Sums

- Given a partition P of $[a,b]$, $P = \{a = x_0, x_1, x_3, \ldots, x_n = b\}$, and $\Delta x_i = x_i - x_{i-1}$ the width of the ith subinterval, $1 \leq i \leq n$;
- Let f be defined on $[a,b]$.
- Then the Right Riemann Sum is

$$\sum_{i=1}^{n} f(x_i)\Delta x_i,$$

and the Left Riemann Sum is

$$\sum_{i=0}^{n} f(x_i)\Delta x_i.$$

The Definite Integral

- Let P be a partition of the interval $[a, b]$, $P = \{x_0, x_1, x_2, ..., x_n\}$ with $a = x_0 \leq x1 \leq x2 \ldots x_n = b$.

- Let $\Delta x_i = x_i - x_{i+1}$ be the width of the ith subinterval, $1 \leq i \leq n$. Let f be a function defined on $[a, b]$.

- We say that f is Riemann integrable on $[a, b]$ if there exists a number Φ such that $L(P, f) \leq \Phi \leq U(P, f)$ for all partitions of $[a, b]$. We write the number as

$$\Phi = \int_a^b f(x)dx$$

and call it the definite integral of f over $[a, b]$.

Theorem 1: If f is Riemann integrable on $[a, b]$, then

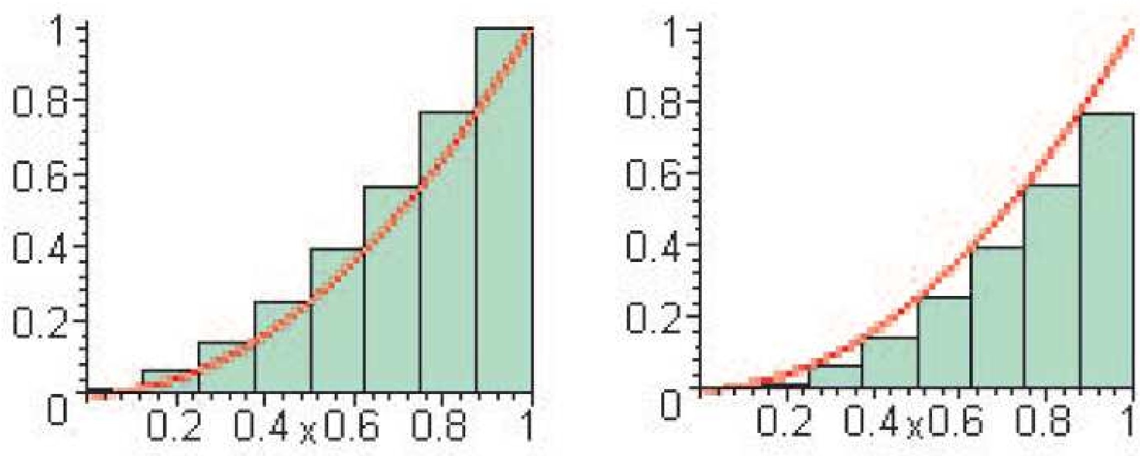
$$\int_a^b f(x)dx = \lim_{\substack{n\to\infty \\ ||P||\to 0}} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

where c_i is any point in the subinterval $[x_{i-1}, x_i]$, and $||P||$ is the maximum length of the Δx_i.

Theorem. *If f is continuous on $[a, b]$, then f is Riemann integrable on $[a, b]$.*

Example

Use an Upper Riemann Sum and a Lower Riemann Sum, first with 8, then with 100 subintervals of equal length to approximate the area under the graph of $y = f(x) = x^2$ on the interval $[0, 1]$.



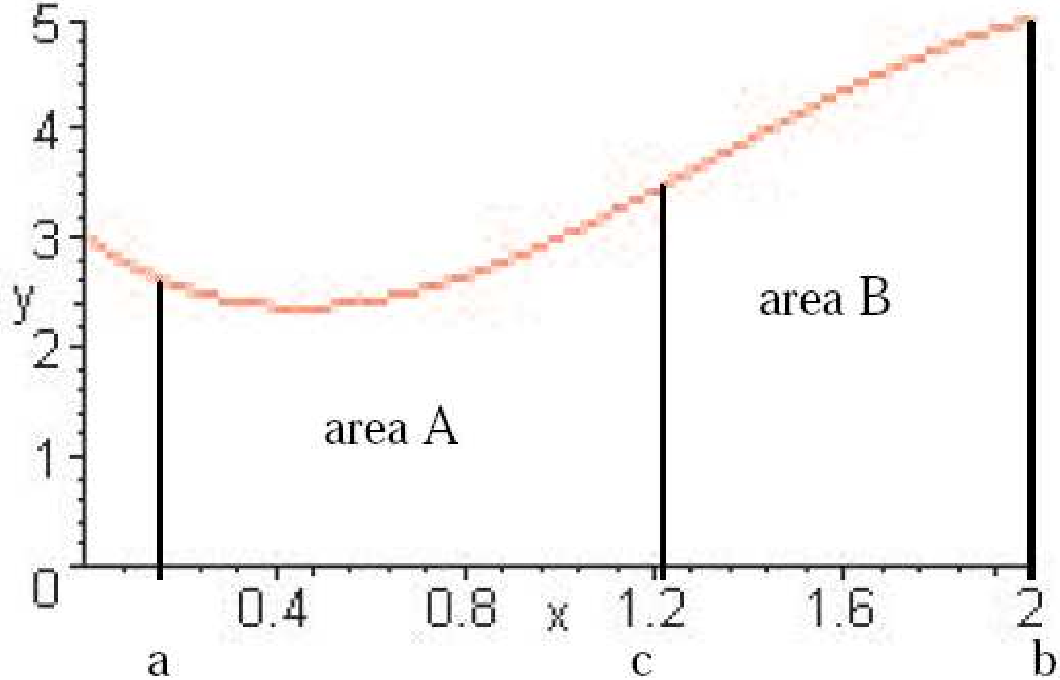
Properties of the Definite Integral

- $\int_a^b f(x)dx = 0$.

- If f is integrable and $f(x) \geq 0$ on $[a, b]$, then $\int_a^b f(x)dx$ equals the area of the region under the graph of f and above the interval $[a, b]$. If $f(x) \leq 0$ on $[a, b]$, then $\int_a^b f(x)dx$ equals the negative of the area of the region between the interval $[a, b]$ and the graph of f.

- **Definition:** $\int_b^a f(x)dx = -\int_a^b f(x)dx$.

$\int_a^c f(x)\, dx + \int_c^b f(x)\, dx = \int_a^b f(x)\, dx.$



- If f and g are integrable on $[a, b]$, then

$$\int_a^b (Af(x) + Bg(x))dx = A\int_a^b f(x)dx + B\int_a^b g(x)dx,$$

for any constants A and B.

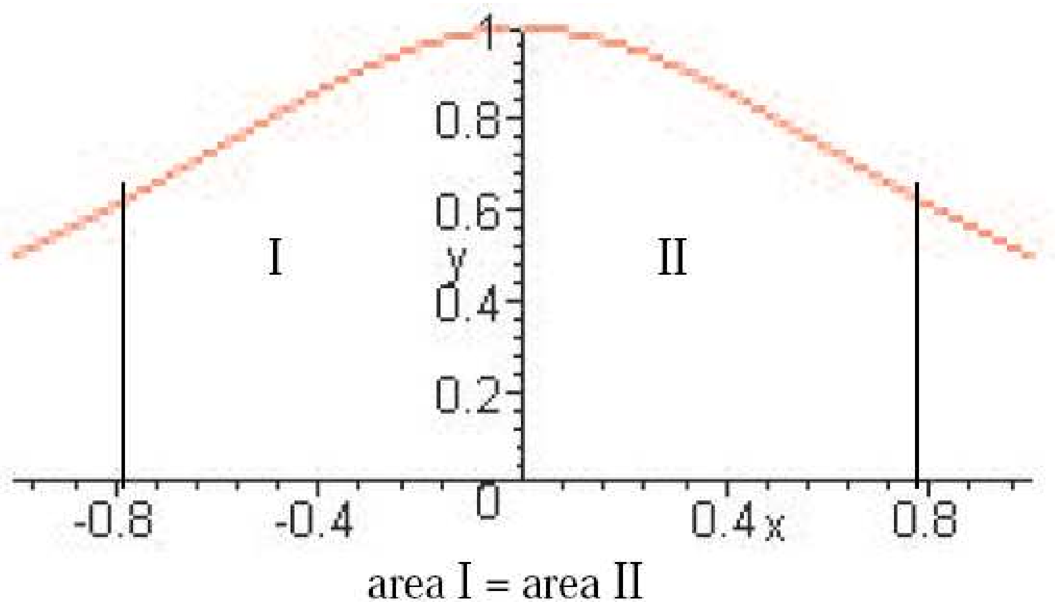
- If f is an odd function, then $\backslash$

$$\int_{-a}^a f(x)dx = 0.$$

That is, the definite integral of an odd function over a symmetric interval is zero.

- If f is an even function, then

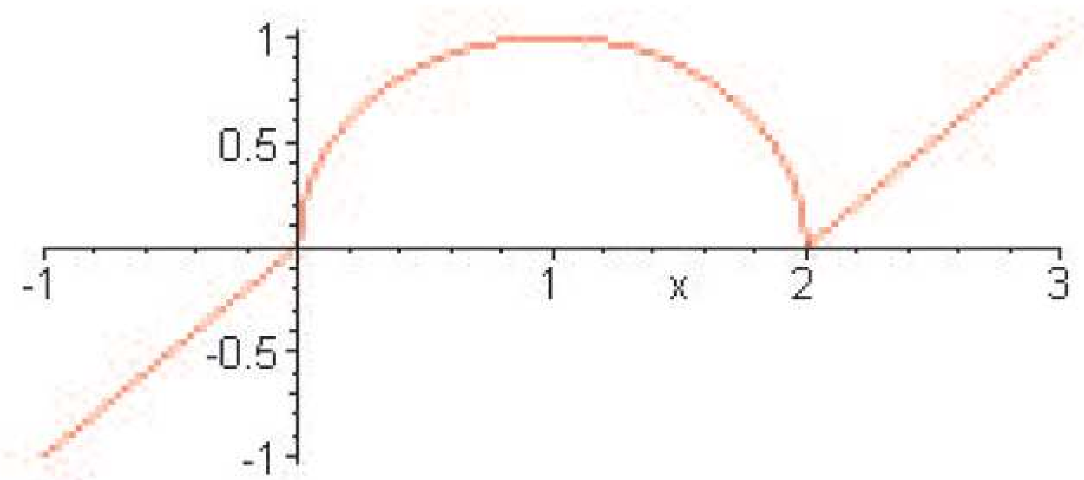
$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx.$$



Example

Let the function f be defined piecewise by

$$f(x) = \begin{cases} x & \text{if } x < 0 \\ \sqrt{-x^2 + 2x} & 0 \leq x \leq 2 \\ x - 2 & \text{otherwise} \end{cases}$$



Mean Value Theorem for Definite Integrals

Theorem. *Let f be continuous on the interval $[a, b]$. Then there exists c in $[a, b]$ such that*

$$\int_a^b f(x)dx = (b - a)f(c).$$

Definition. *The* average value *of a continuous function on the interval $[a, b]$ is*

$$\frac{1}{b - a}\int_a^b f(x)dx.$$