Arc Length

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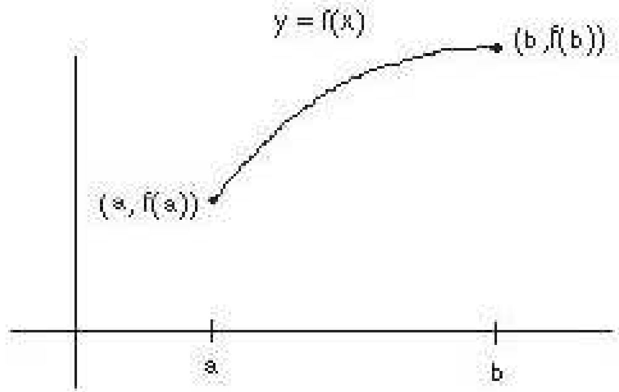
- Another illustration of the Riemann Sum modeling method, consider the problem of computing the length of a curve in the plane.

- We will assume that $y = f(x)$ is a continuous function defined on the interval $[a, b]$ and that $f'(x)$ exists at every point of the interval.

- How to determine the length of the graph of f from the point $(a, f(a))$ to the point $(b, f(b))$.


y = f(x)
(b ,f(b))
(a, f(a))
a
b

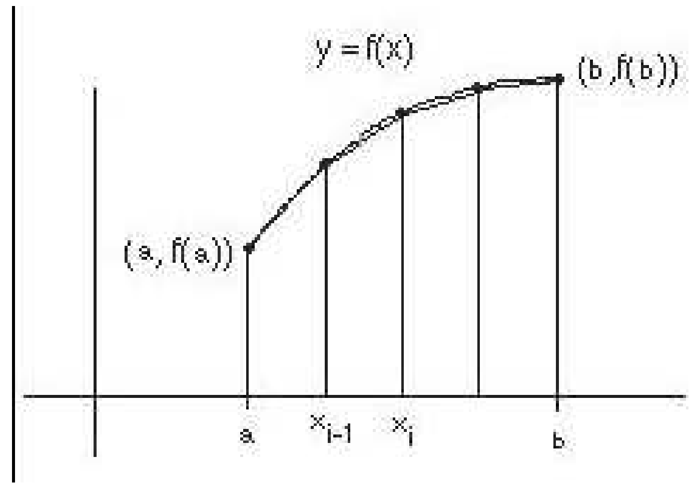
Summary of the Riemann Sum Method for Arc Length

- Divide the interval $[a, b]$ into n subintervals of equal length $x = (b-a)/n$. Call the points of the subdivision $a = x_0 \leq x_1 \leq x_2 \leq x_3 \Delta\Delta\Delta x_{n-1} \leq x_n = b$, where $x_i = a + i\Delta x$ for each i.

- On each subinterval $[x_{i-1}, x_i]$ connect the points $(x_{i-1}, f(x_{i-1})$ and $(x_i, f(x_i)$ on the graph of f with straight lines.

- The length of the straight-line segment connecting the two points is

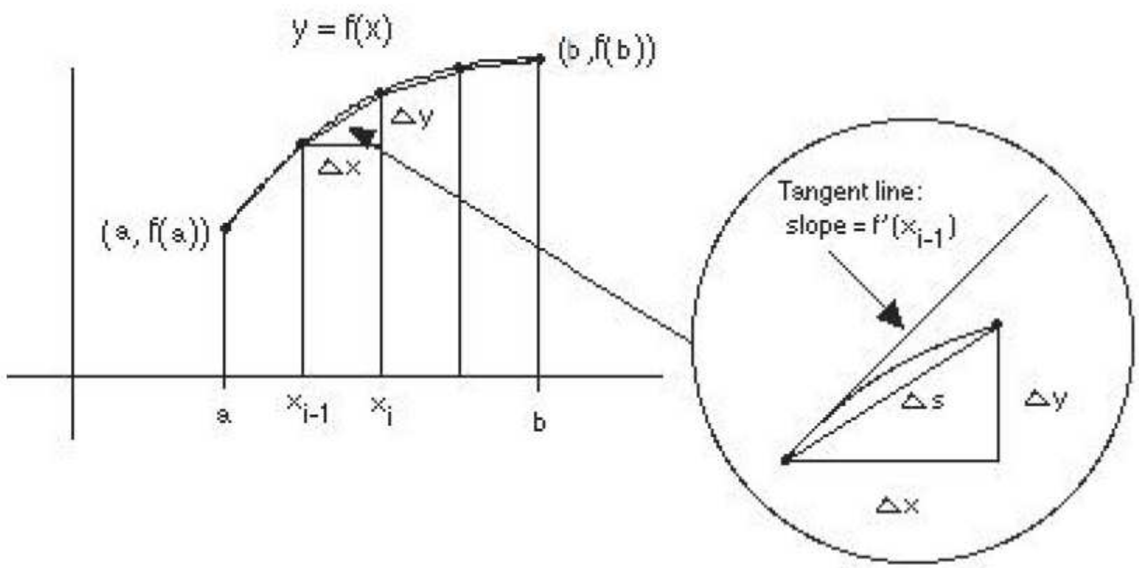
$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2.$$


y = f(x)
(b, f(b))
(a, f(a))
a
x_{i-1}
x_i
b

Summary ...

$$\Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$\Delta s = \sqrt{(\Delta x)^2 \left(1 + \frac{(\Delta y)^2}{(\Delta x)^2}\right)} = \Delta x \sqrt{1 + \frac{(\Delta y)^2}{(\Delta x)^2}}$$



- We can replace Δs by the approximate value

$$\Delta s \simeq \Delta x \sqrt{1 + [f'(x_{i-1})]^2}.$$

- The sum of the approximate lengths of these line segments provides an approximation to the length of the curve

$$\sum_{i=1}^{n} \sqrt{1 + [f'(x_{i-1})]^2} \Delta x.$$

- Taking the limit as $x \to 0$, the above approximation approaches the length of the curve. The limit is

$$L = \lim_{\Delta x \to 0} \sum_{i=1}^{n} \sqrt{1 + [f'(x_{i-1})]^2} \Delta x = \int_a^b \sqrt{1 + [f'(x)]^2} dx.$$

The Arc Formula

- The integral formula to compute the length L of the graph of f between $x = a$ and $x = b$ is

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} dx.$$

Example

- Find the length of the arc $y = x^{3/2}$, from $x = 0$ to $x = 1$.

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$$\begin{aligned} L &= \int_0^1 \sqrt{1 + \left[\frac{3}{2}x^{1/2}\right]^2}\, dx \\ &= \int_0^1 \sqrt{1 + \left[\frac{9}{4}x\right]}\, dx \end{aligned}$$

- Find the length of the curve $y = x^4 + \frac{1}{32x^2}$ from x = 1 to x = 2.

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$$y' = 4x^3 - \frac{2}{32x^3} = 4x^3 - \frac{1}{16x^3}$$

$$\begin{aligned}
L &= \int_1^2 \sqrt{1 + \left[4x^3 - \frac{1}{16x^3}\right]^2}\, dx \\
&= \int_1^2 \sqrt{1 + 16x^6 - \frac{8}{16} + \frac{1}{256x^6}}\, dx \\
&= \int_1^2 \sqrt{\frac{8}{16} + 16x^6 + \frac{1}{256x^6}}\, dx \\
&= \int_1^2 \sqrt{\left(4x^3 + \frac{1}{16x^3}\right)^2}\, dx \\
&= \int_1^2 \left(4x^3 + \frac{1}{16x^3}\right) dx \\
&= \left(x^4 - \frac{1}{32x^2}\right)\Bigg|_1^2 \\
&= 15 + \frac{3}{128}
\end{aligned}$$