Case Study: Flood Watch

11/18/2005

- If we are given the continuous derivative f' of a function on the interval $[a, b]$, then one version of the Fundamental Theorem of Calculus states that

$$\int_a^b f'(x)dx = f(b) - f(a).$$

- The derivative $f'(x)$ may be given at only a finite number of points of the interval.

- We will use the Riemann sum

$$\sum_{i=1}^{n} f'(x_i)\Delta x_i.$$

Objective

- From data collected ten years apart, to determine if the Gorge River has an increased or decreased likelihood of flooding.

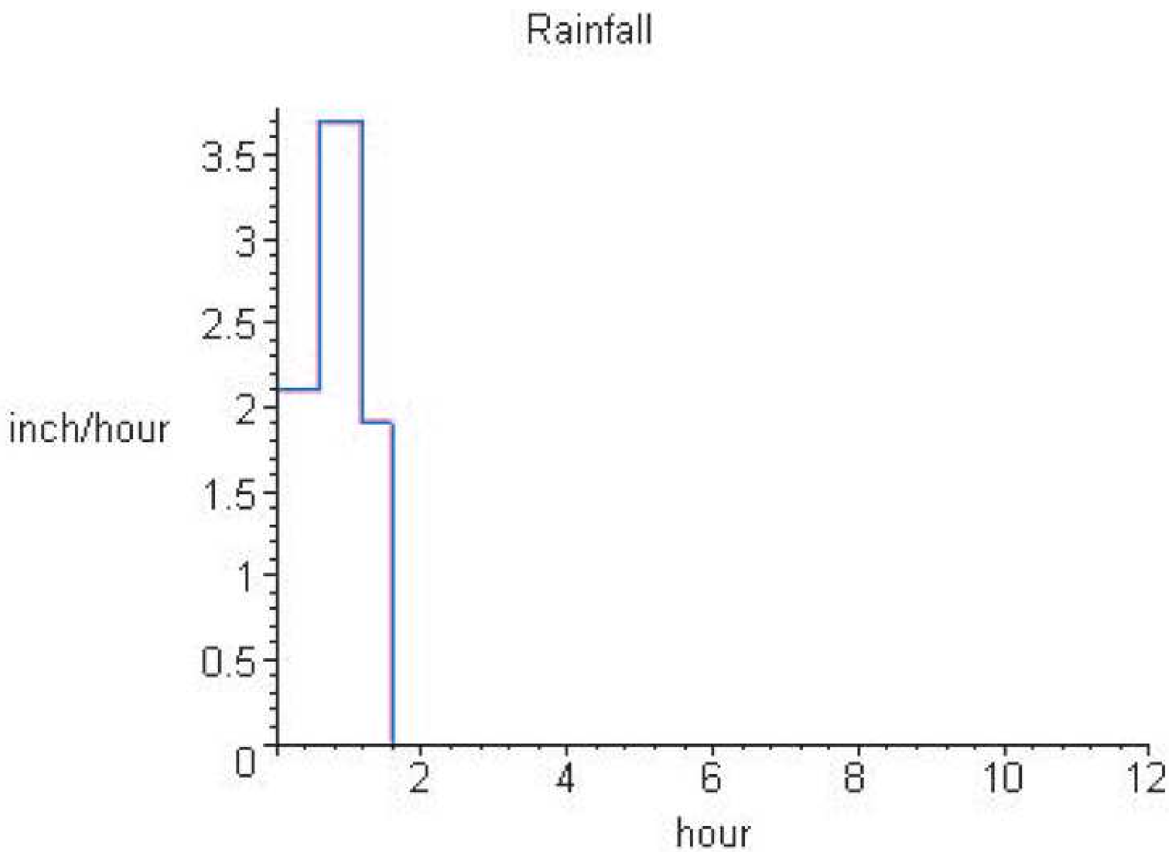
River Flooding: Background

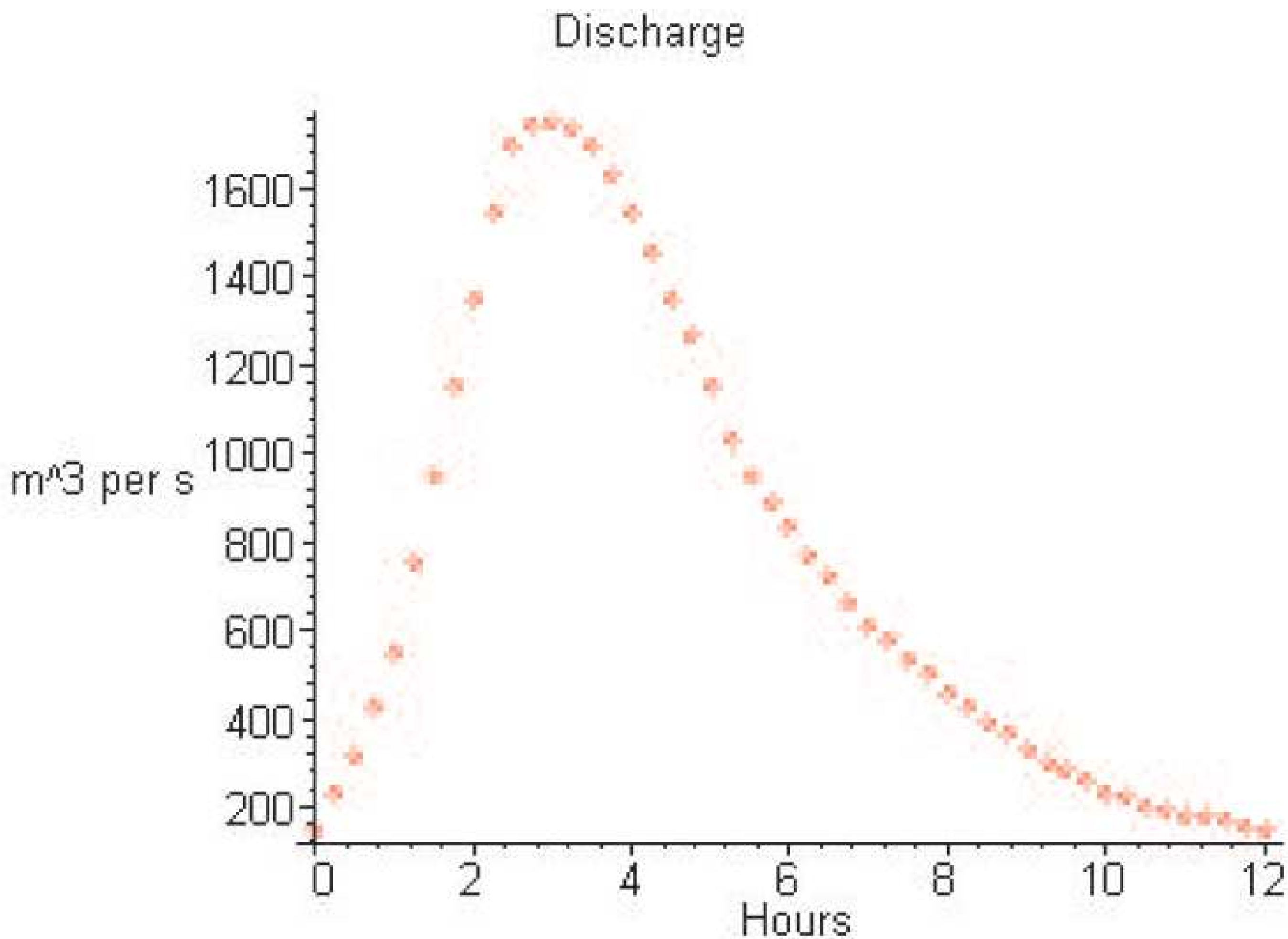
- The fundamental way to display the observations of stream discharge is a *hydrograph*.

- A hydrograph has two components: the amount of rainfall (depth per unit time) and discharge (volume of water per unit time).

- Prior to a rain storm, the stream will be flowing at some background level of discharge known as base flow.

Example of a Hydrograph





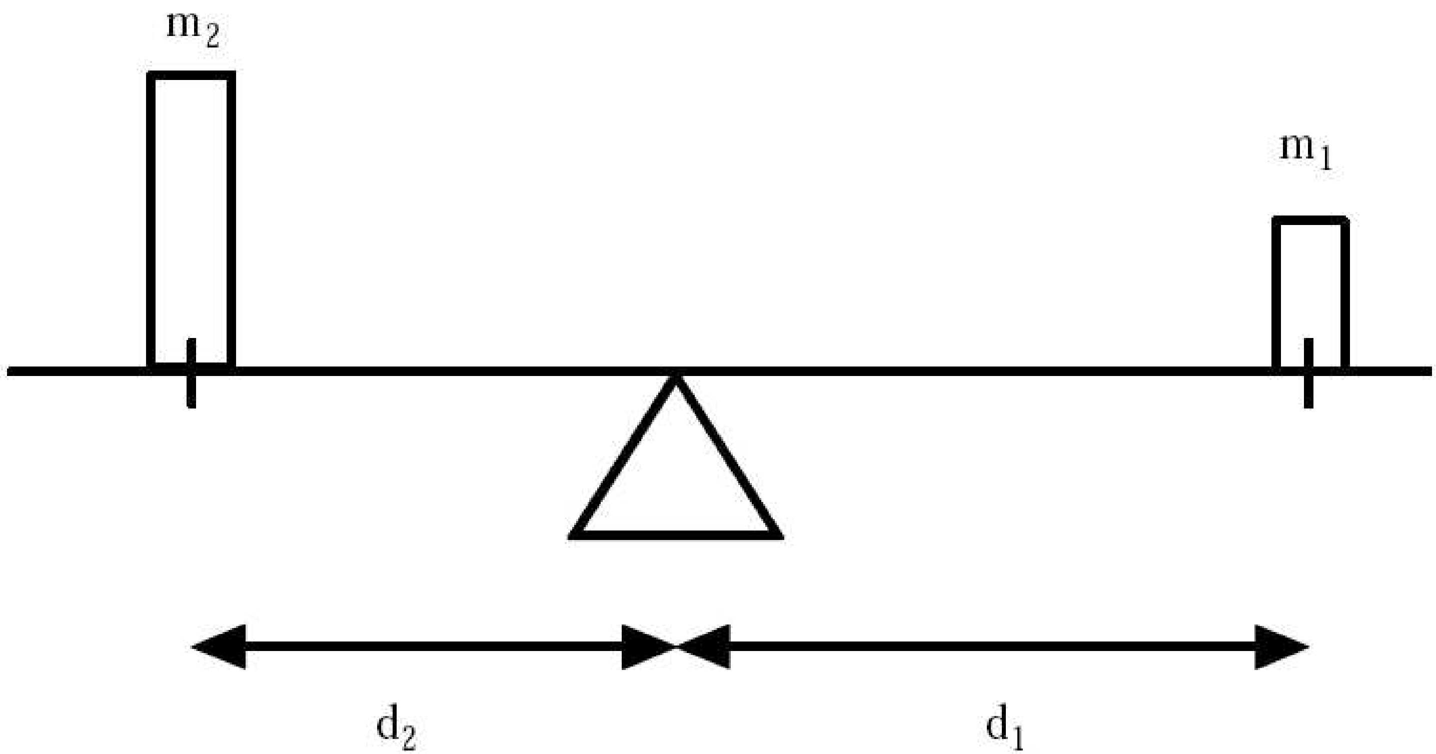
Discharge
1600
1400
1200
1000
800
600
400
200
m^3 per s
0
2
4
6
8
10
12
Hours

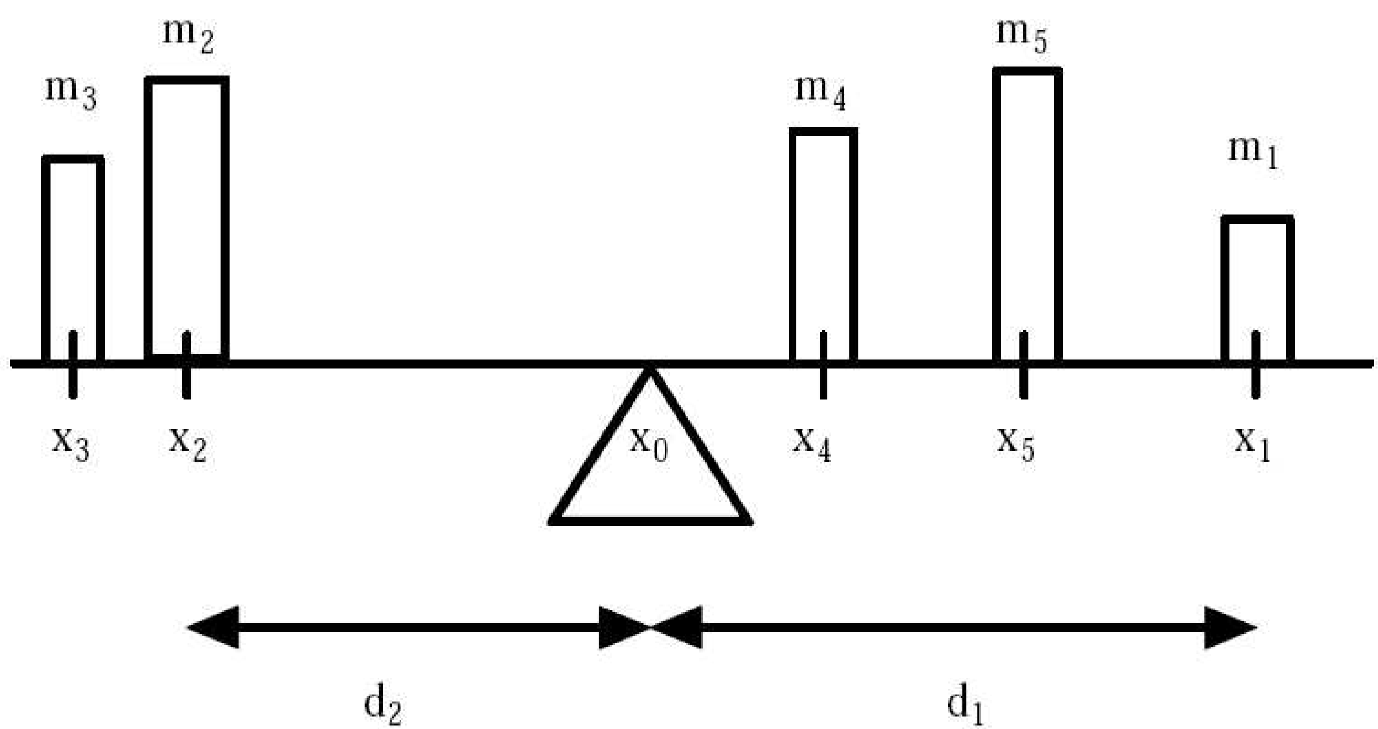
Theorem: Suppose the region of area A is defined as lying between two curves $y = f(x)$ and $y = g(x)$ where $f(x) \geq g(x)$ and $a \leq x \leq b$. Then its centroid is located at the point (x_0, y_0) given by

$$x_0 = \frac{1}{A} \int_a^b x(f(x) - g(x))\, dx$$

$$y_0 = \frac{1}{A} \int_a^b \frac{f(x)^2 - g(x)^2}{2}\, dx$$

Derivation of Formulas for the Centroid





m2
m3
m4
m5
m1
x3
x2
x0
x4
x5
x1
d2
d1

- Suppose now that we consider n masses $m_1, m_2, m_3, ..., m_n$ located on a see-saw at the points $x_1, x_2, x_3, ..., x_n$, and suppose further that the pivot point is located at x_0.

- The balance is achieved when

$$\sum_{k=1}^{n} m_k(x_k - x_0).$$

- Calling $\sum_{k=1}^{n} m_k$ the *total mass*, we then have that

$$x_0 = \frac{\sum_{k=1}^{n} m_k x_k}{\text{total mass}}.$$

- If the rectangles pictured above are simply two-dimensional regions

$$x_0 = \frac{\sum_{k=1}^{n} x_k A_k}{\text{total area}}.$$

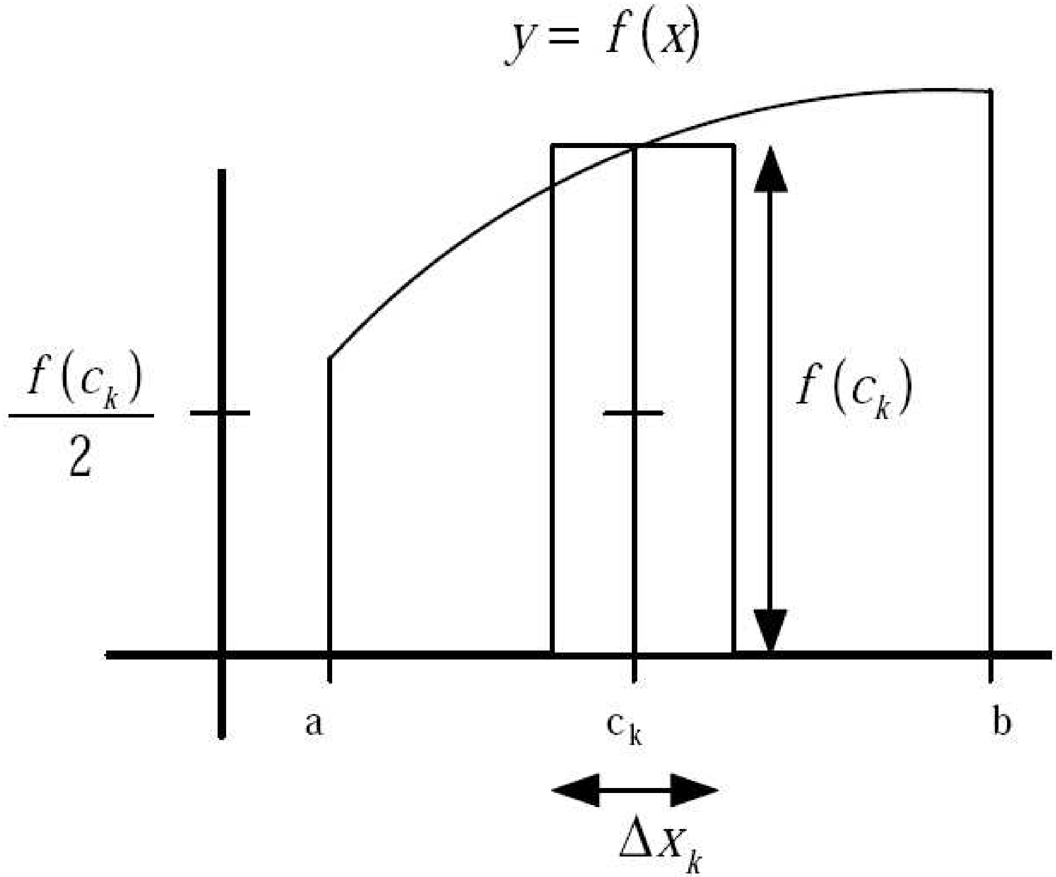
- The product $x_k A_k$ is called the moment of the kth rectangle.

- Suppose now that we start with a function $y = f(x)$ defined on an interval $[a, b]$.

- We partition the interval into n subintervals with the points $x_0 < x_1 < x_2 < \ldots < x_n$.

- Let c_k be the midpoint of the kth subinterval.

- Then $c_k A_k = c_k f(c_k)\Delta x_k$ is the moment of the kth rectangle with respect to the y-axis.


$y = f(x)$
$\frac{f(c_k)}{2}$
$f(c_k)$
a
c_k
b
Δx_k

- The sum of the moments of the n rectangles is a Riemann sum whose limit is the definite integral (assumed to exist) of the function over the interval $[a, b]$:

$$\lim_{n \to \infty} \sum_{k=1}^{n} c_k f(c_k) \Delta x_k = \int_a^b x f(x) dx.$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{f(c_k)}{2} f(c_k)\Delta x_k = \frac{1}{2}\int_a^b f(x)^2\,dx$$

Finally, the point (x_0, y_0) whose coordinates satisfy

$$Ax_0 = \int_a^b x f(x)\,dx$$

$$Ay_0 = \frac{1}{2}\int_a^b f(x)^2\,dx$$

where A is the area under the graph of f, is called the *centroid* of the region.

Back to Example of a Hydrograph

- The area of the rainfall data is

$$A = .6(2.1) + .6(3.7) + .4(1.9) = 4.24.$$

- The coordinates of the centroid of the rainfall data are:

$$\begin{aligned} x_0 &= \frac{1}{A}(.3(2.1)(.6) + .9(3.7)(.6) + 1.4(1.9)(.4)) \simeq .81132 \\ y_0 &= \frac{1}{A}(2.1^2)(.6) + (3.72)(.6) + (1.92)(.4) \simeq 1.45094 \end{aligned}$$

$$A = 3600 \sum_{k=1}^{n} (f(c_k) - y_{\text{min}}) \Delta x_k \approx 7418$$

$$x_0 = \frac{1}{A} \sum_{k=1}^{n} c_k (f(c_k) - y_{\text{min}}) \Delta x_k \approx 4.294$$

$$y_0 = \frac{1}{2A} \sum_{k=1}^{n} (f(c_k)^2 - y_{\text{min}}^2) \Delta x_k \approx 691.232$$