Limit of a Function

10/05/2005

The Legacy of Galileo, Newton, and Leibniz

- Galileo
  - was interested in falling bodies.
  - forged a new scientific methodology - *observe nature, construct experiments to test what you observe, and construct theories that explain the observations*.

- Newton
  - was able, using his new tools of calculus, to explain why falling bodies behave in this way: an object, falling under the influence of gravity, will have constant acceleration of $9.8m/sec^2$.
  - his laws of motion and of universal gravitation drew under one simple mathematical theory Newton's laws of falling bodies, Kepler's laws of planetary motion, the motion of a simple pendulum, and virtually every other instance of dynamic motion observed in the universe.

- Leibnitz
  - co-inventor of calculus, took a slightly different point of view but also studied rates of change in a general setting.

Newton's Question

- How do we find the velocity of a moving object at time t?

- What in fact do we mean by *velocity* of the object at the instant of time t?? We know how to find the average velocity of an object during a time interval $[t_1, t_2]$?

The average velocity during a time interval is the distance traveled divided by the elapsed time, i.e.

$$\text{Average velocity over}[t_1, t_2] = \frac{\text{distance traveled}}{t_2 - t_1}.$$

Definition

Let $x(t)$ be a function that gives the position at time t of an object moving on the x-axis. Then

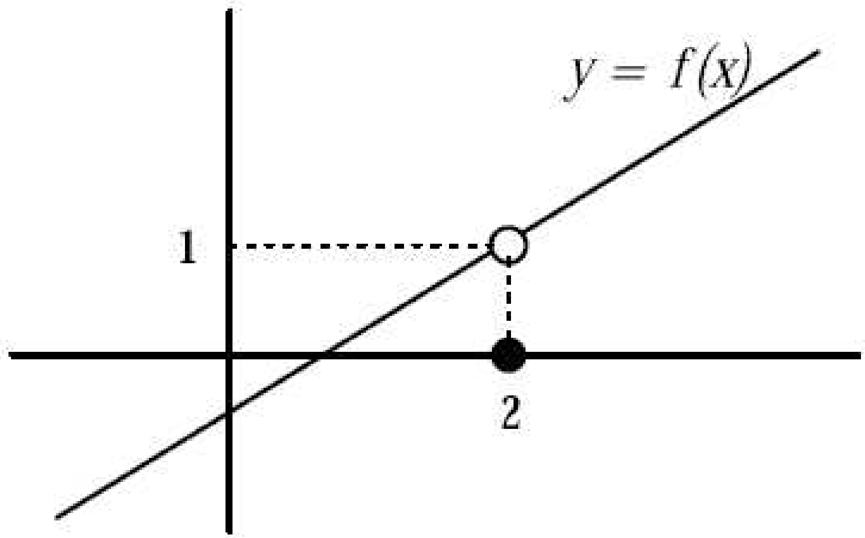
$$\text{Ave vel}[t_1, t_2] = \frac{x(t_2) - x(t_1)}{t_2 - t_1}$$

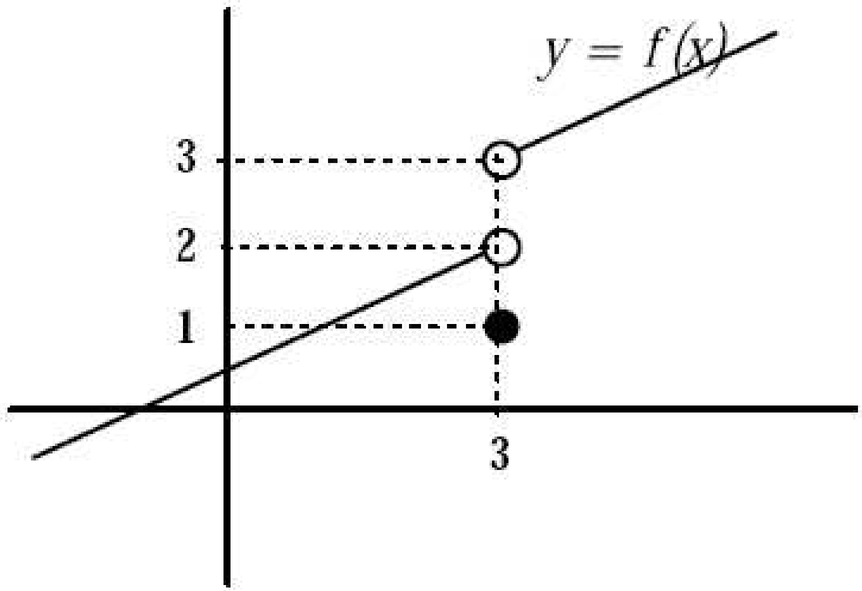
$$\text{Velocity}(t) = \lim_{h \to 0} \frac{x(t+h) - x(t)}{h}.$$

Limit of a Function – Definition

We say that a function f approaches the limit L as x approaches a, written $\lim_{x \to a} f(x) = L$, if we can make $f(x)$ as close to L as we please by taking x sufficiently close to a.

Example





$y = f(x)$
3
2
1
3

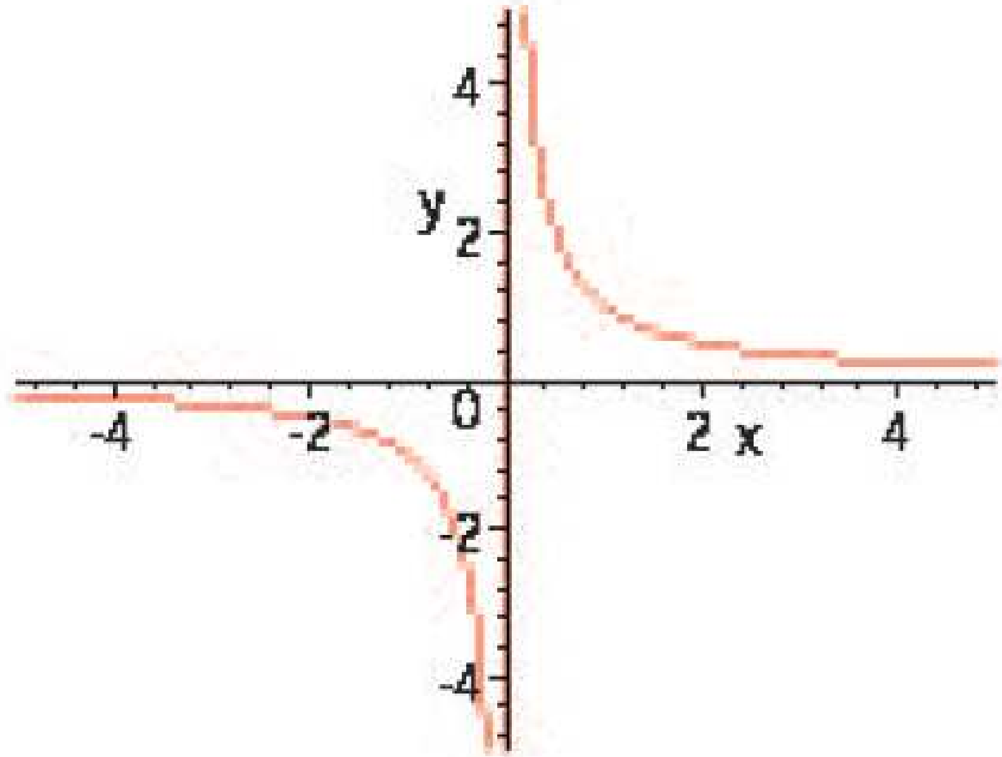
Theorem. *The limit of f as $x \to a$ exists if and only if both the right-hand and left-hand limits exist and have the same value. I.e.*

$$\lim_{x \to a} f(x) = L \Leftrightarrow \lim_{x \to a^-} f(x) = L \text{ and } \lim_{x \to a^+} f(x) = L.$$

Examples

Compute the limits:

- $\lim_{x\to 2}\frac{x-2}{x+3}$
- $\lim_{x\to -1}\frac{x^2-1}{x-1}$
- $\lim_{x\to 0}\frac{1}{x}$


4
y
2
0
-2
-4
-4
-2
2
x
4

Theorem. *If* $\lim_{x\to a} f(x) = A$ *and* $\lim_{x\to a} g(x) = B$ *both exist, then*

1. $\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = A+B$
2. $\lim_{x\to a}(f(x)-g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = A-B$
3. $\lim_{x\to a}(f(x)g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = A\cdot B$
4. $\lim_{x\to a}(f(x)/g(x)) = \lim_{x\to a} f(x)/ \lim_{x\to a} g(x) = A/B$ $(B \neq 0)$.

Examples

1. $\lim_{x-1} \frac{x^2-2x+3}{x^3+3x-1}$

2. $\lim_{x\to 0} \frac{|x|}{x}$

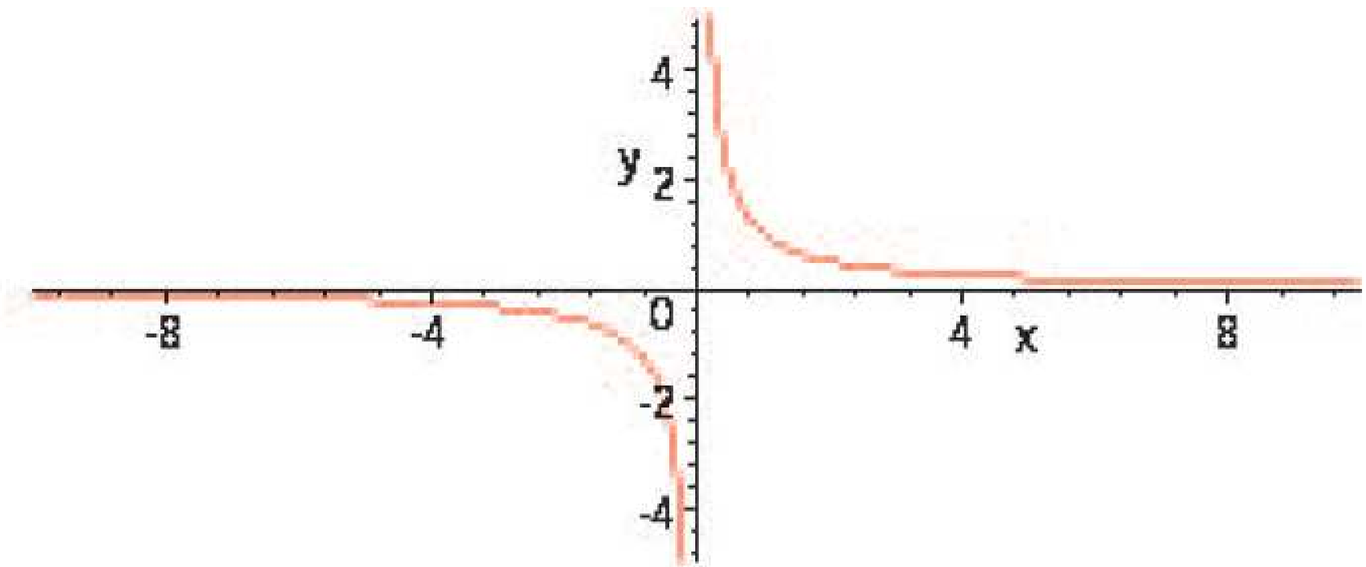
3. Let $f(x) = 1/x$. Compute $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

Limits at Infinity

$\lim_{x\to\infty} f(x) = L$ means that the value of $f(x)$ approaches L as the value of x approaches $+\infty$. This means that $f(x)$ can be made as close to L as we please by taking the value of x sufficiently large. Similarly, $\lim_{x\to\infty} f(x) = L$ means that $f(x)$ can be made as close to L as we please by taking the value of x sufficiently small (in the negative direction).

Example

$\lim_{x\to\infty} 1/x = 0.$



More Examples

Evaluate the limits:

1. $\lim_{x\to\infty} \frac{x-1}{x^3+2}$

2. $\lim_{x\to\infty} \frac{3x^2-2x+1}{4x^2-1}$

3. $\lim_{x\to\infty} \frac{x^4-x^2+2}{x^3+3}$

Dominant Term Rue

For the limit $\lim_{x\to\infty} P(x)/Q(x)$, where $P(x)$ is a polynomial of degree n and $Q(x)$ is a polynomial of degree m,

1. If $n < m$, the limit is 0,

2. If $n > m$, the limit is $\pm\infty$,

3. If $n = m$, the limit is the quotient of the coefficients of the highest powers.

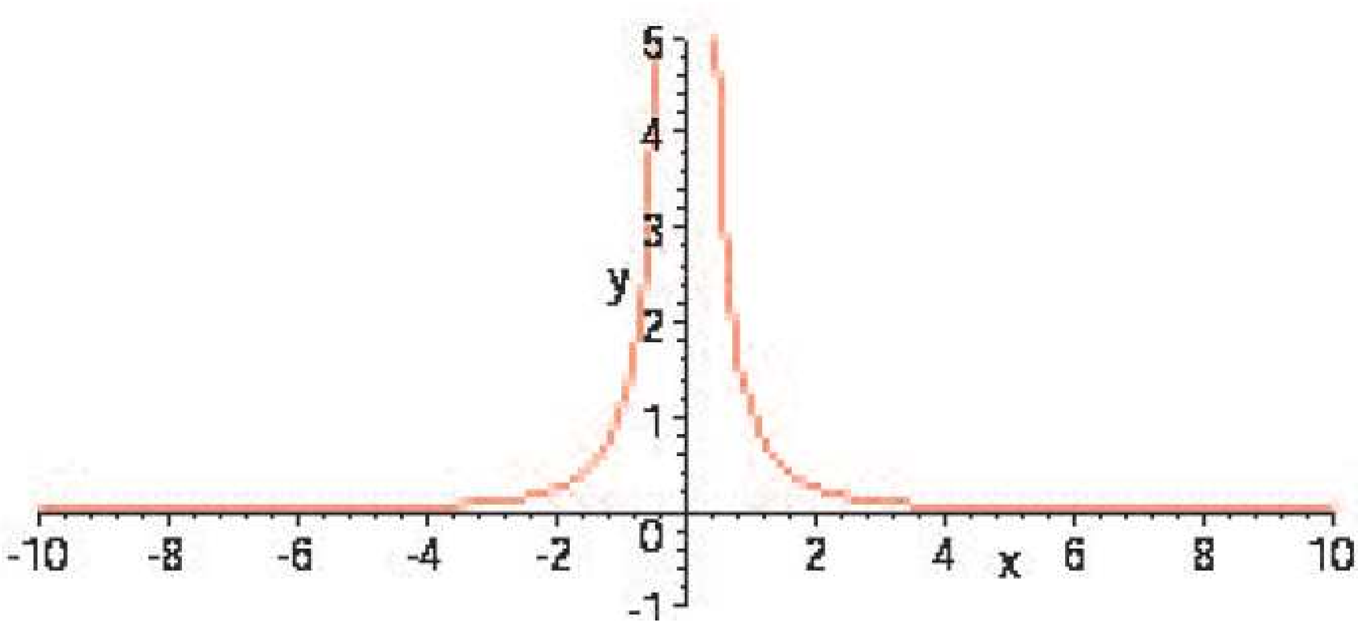
Example

Evaluate the limit:

$$\lim_{x\to\infty} \frac{x}{\sqrt{3x^2+2}}$$

Infinite Limits

Compute the limit $\lim_{x \to 0} 1/x^2$.



Evaluate $\lim_{x\to\pi/2} \tan x$.