The Derivative

October 10, 2005

Definition

- The derivative of a function f is a new function defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

- We will say that a function f is differentiable at a point $x = a$ if the derivative function f' exists at a.

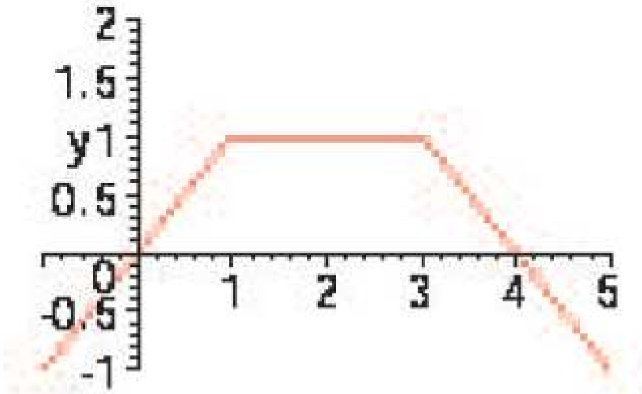
Example

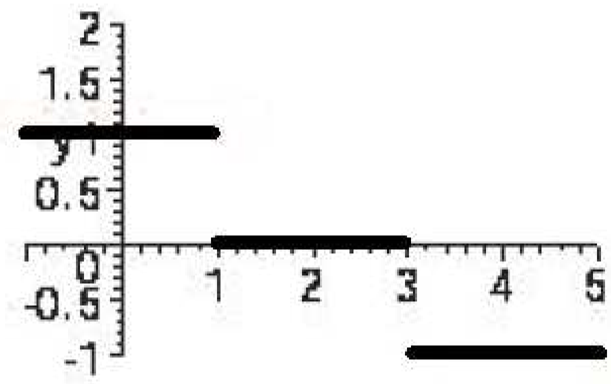
Suppose we consider the piecewise defined function

$$f(x) = \begin{cases} x & x \leq 1 \\ 1 & 1 < x < 3 \\ -x + 4 & 3 \leq x \end{cases}$$

It's derivative is:

$$f(x) = \begin{cases} 1 & x < 1 \\ 0 & 1 < x < 3 \\ -1 & 3 < x \end{cases}$$


2
1.5
y1
0.5
0
-0.5
-1
1
2
3
4
5


2
1.5
y
0.5
0
-0.5
-1
1
2
3
4
5

Example

$$f(x) = k,$$

where k is a constant.

Example

$$f(x) = ax + b,$$

a, b constants.

The derivative of x^2

- For $f(x) = x^2$, we have

$$f'(x) = 2x$$

The derivative of x^3

- For $f(x) = x^3$, we have

$$f'(x) = 3x^2$$

The derivative of $1/x$

- For $f(x) = \frac{1}{x}$, we have

$$f'(x) = -\frac{1}{x^2}$$

The derivative of $\sqrt{x}$

- For $f(x) = \sqrt{x}$, we have

$$f(x) = \frac{1}{2\sqrt{x}}$$

The Power Rule

- Suppose that $f(x) = x^r$, where r is any real number. Then

$$f'(x) = rx^{r-1}.$$

Example

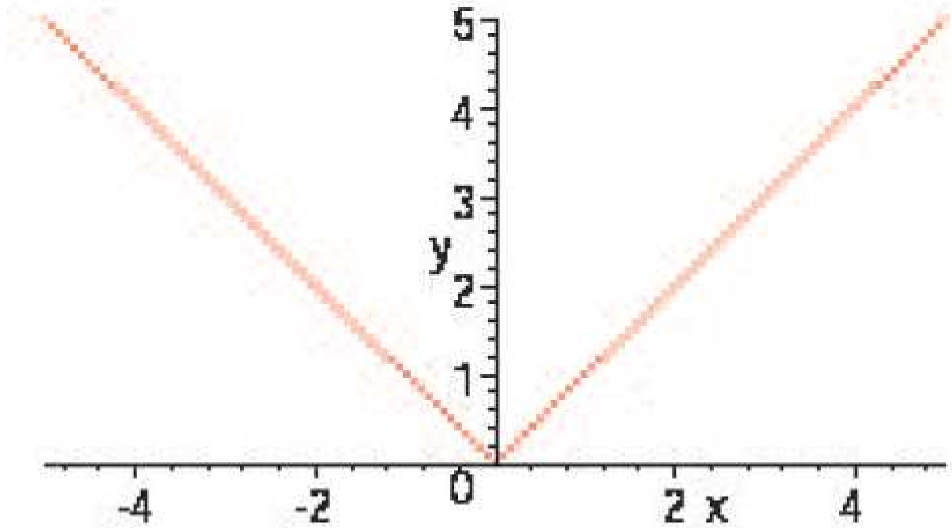
- Find an equation of the tangent line to the graph of $f(x) = x^{4/3}$ at the point where $x = 1$.

$$y = f(1) + f'(1)(x - 1).$$

Example

- Find the derivative of $f(x) = |x|$.

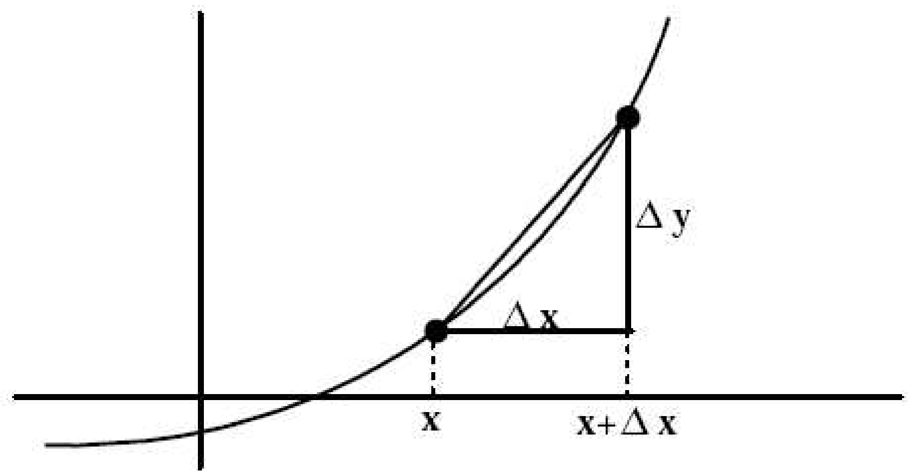
$$\begin{aligned} \lim_{h \to 0^+} \frac{|0+h| - 0}{h} &= 1 \\ \lim_{h \to 0^-} \frac{|0+h| - 0}{h} &= -1 \end{aligned}$$


5
4
3
y
2
1
0
-4
-2
2 x
4

Notation for the Derivative

$$y' = D_x y = \frac{dy}{dx} = \frac{d}{dx} f(x) = f'(x).$$

The notation $\frac{dy}{dx}$



Example

For the function $y = f(x) = 1/x$, find the slope of its tangent line at $x = 2$. Compare it with the average rate of change over the interval $[2, 3]$.

Higher Order Derivatives

- When we differentiate a function $f(x)$ we obtain a new function $f'(x)$.
- The derivative is again a candidate for differentiation, and we call its derivative *the second derivative* of $f(x)$.
- So long as the derivatives exist we can continue this process to obtain a succession of higher derivatives.

$$y'' = f''(x) = \frac{d^2y}{d^2x} = \frac{d}{dx}\frac{d}{dx}f(x) = \frac{d^2}{dx^2}f(x) = Dx^2y = Dx^2f(x).$$

The nth derivative, where n is a positive integer

$$y^{(n)} = f^{(n)}(x) = \frac{d^n y}{d^n x} = \frac{d^n}{dx^n} f(x) = Dx^n y = Dx^n f(x).$$