2.10 The Mean Value Theorem and 2.11 Implicit Differentiation

Mathematics 3
Lecture 11
Dartmouth College

January 27, 2010

Derivatives of the Trig Functions (cont'd)

Before we begin, let's practice an example problem involving trig functions...

Example 1: Find the derivative of the function

$$y = f(x) = \sqrt{\cos(x^2 - 5x + 3)}.$$

by using Leibniz notation for the derivatives involved.

Derivatives of the Trig Functions (cont'd)

Before we begin, let's practice an example problem involving trig functions...

Example 1: Find the derivative of the function

$$y = f(x) = \sqrt{\cos(x^2 - 5x + 3)}.$$

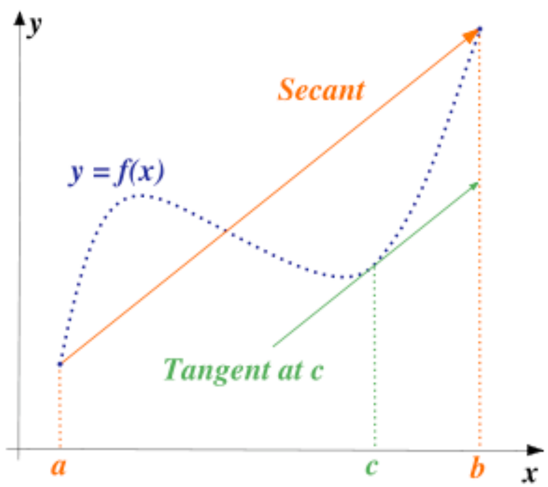
by using Leibniz notation for the derivatives involved.

Answer: $\frac{dy}{dx} = \frac{(5x-2)\sin(x^2-5x+3)}{2\sqrt{\cos(x^2-5x+3)}}$

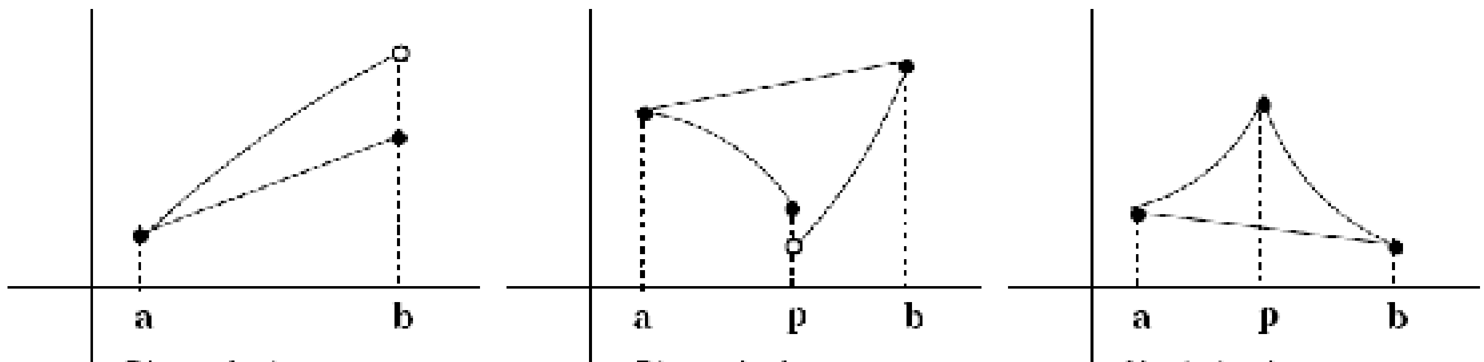
The Mean Value Theorem

Theorem 1. $(p.161)$ *Suppose that f is defined and continuous on a closed interval $[a, b]$, and suppose that f' exists on the open interval (a, b). Then there exists a point c in (a, b) such that*

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$



Examples where the Mean Value Theorem fails...



Discontinuity at an end point

Discontinuity at an interior point p

No derivative at an interior point p

Example 2: Does the function

$$f(x) = x^{\frac{2}{3}}$$

satisfy the Mean Value Theorem on $[-1, 1]$. Why or why not?

Example 3

Verify the conclusion of the Mean Value Theorem for the function

$$f(x) = x^3 + 3x^2 + 3x$$

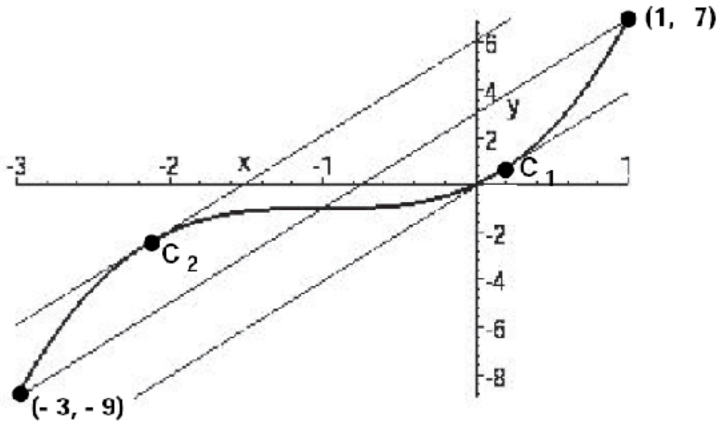
on the interval $[-3, 1]$.

Example 3

Verify the conclusion of the Mean Value Theorem for the function

$$f(x) = x^3 + 3x^2 + 3x$$

on the interval $[-3, 1]$.



Monotonicity of Functions on Intervals

Recall: An **interval** I is a set of real numbers lying between a and b, where a and b are real numbers or $\pm\infty$ (and may/may not include a or b, e.g., $I = [0, 5), (-\infty, 6), [-1, 2]$, etc.

Suppose that the function f is defined on an interval I, and let x_1 and x_2 denote points in I:

1. f is increasing on I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.
2. f is decreasing on I if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.
3. f is nondecreasing on I if $f(x_1) \leq f(x_2)$ whenever $x_1 < x_2$.
4. f is nonincreasing on I if $f(x_1) \geq f(x_2)$ whenever $x_1 < x_2$.

Testing Monotonicity via Derivatives

Theorem. *Let I be an interval and let J be the open interval consisting of I minus its endpoints (if any). Suppose that f is continuous on I and differentiable on J. Then*

1. If $f'(x) > 0$ for every $x \in J$, then f is increasing on I.

2. If $f'(x) < 0$ for every $x \in J$, then f is decreasing on I.

3. If $f'(x) \geq 0$ for every $x \in J$, then f is nondecreasing on I.

4. If $f'(x) \leq 0$ for every $x \in J$, then f is nonincreasing on I.

Example 4

Find the intervals on which the function

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 5$$

is increasing and those on which it is decreasing.

Example 4

Find the intervals on which the function

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 5$$

is increasing and those on which it is decreasing.

Answer: f is increasing on $(-1, 0) \cup (2, +\infty)$ and decreasing on $(-\infty, -1) \cup (0, 2)$.

Example 5

On what interval(s) is the function

$$f(x) = 2x - \sin(x)$$

increasing or decreasing (if any)?

Example 5

On what interval(s) is the function

$$f(x) = 2x - \sin(x)$$

increasing or decreasing (if any)?

Answer: $f'(x) = 2 - \cos(x) > 0$ **always**! $\Rightarrow f$ is always increasing!

The Extreme Value Theorem

Theorem. *If f is continuous on a closed interval $[a, b]$, then there is a point c_1 in the interval where f assumes its maximum value, i.e. $f(x) \leq f(c_1)$ for every x in $[a, b]$, and a point c_2 where f assumes its minimum value, i.e. $f(x) \geq f(c_2)$ for every x in $[a, b]$.*

The Extreme Value Theorem

Theorem. *If f is continuous on a closed interval $[a, b]$, then there is a point c_1 in the interval where f assumes its maximum value, i.e. $f(x) \leq f(c_1)$ for every x in $[a, b]$, and a point c_2 where f assumes its minimum value, i.e. $f(x) \geq f(c_2)$ for every x in $[a, b]$.*

Zen: A continuous function on a closed and bounded interval $[a, b]$ always has extreme values (i.e., max and min) somewhere in the interval. This is an "existence theorem" and is very hard to prove, in generality (Math 35/54/63).

The Extreme Value Theorem

Theorem. *If f is continuous on a closed interval $[a, b]$, then there is a point c_1 in the interval where f assumes its maximum value, i.e. $f(x) \leq f(c_1)$ for every x in $[a, b]$, and a point c_2 where f assumes its minimum value, i.e. $f(x) \geq f(c_2)$ for every x in $[a, b]$.*

Zen: A continuous function on a closed and bounded interval $[a, b]$ always has extreme values (i.e., max and min) somewhere in the interval. This is an "existence theorem" and is very hard to prove, in generality (Math 35/54/63).

Important Question: How do we FIND these extreme values?

Finding Extreme Values with Derivatives

Theorem. *If f is defined in an open interval (a, b) and achieves a maximum (or minimum) value at a point $c \in (a, b)$ where $f'(c)$ exists, then $f'(c) = 0$.*

Zen: An extreme value (max/min) of a differentiable function in an open interval (a, b) must occur where the graph has a horizontal tangent line. But, just because $f'(c) = 0$ does NOT mean you have an extreme value at $x = c$. Moreover, on a closed interval $[a, b]$ we have to also check the endpoints for possible max/mins.

Def: A point $x = c$ where $f'(c) = 0$, or *does not exist*, is called a critical point of the function f.

Example 6

For the function

$$f(x) = 2x^3 - 6x^2 - 18x + 1,$$

let us find the points in the interval $[-4, 4]$ where the function assumes its maximum and minimum values.

Example 6

For the function

$$f(x) = 2x^3 - 6x^2 - 18x + 1,$$

let us find the points in the interval $[-4, 4]$ where the function assumes its maximum and minimum values.

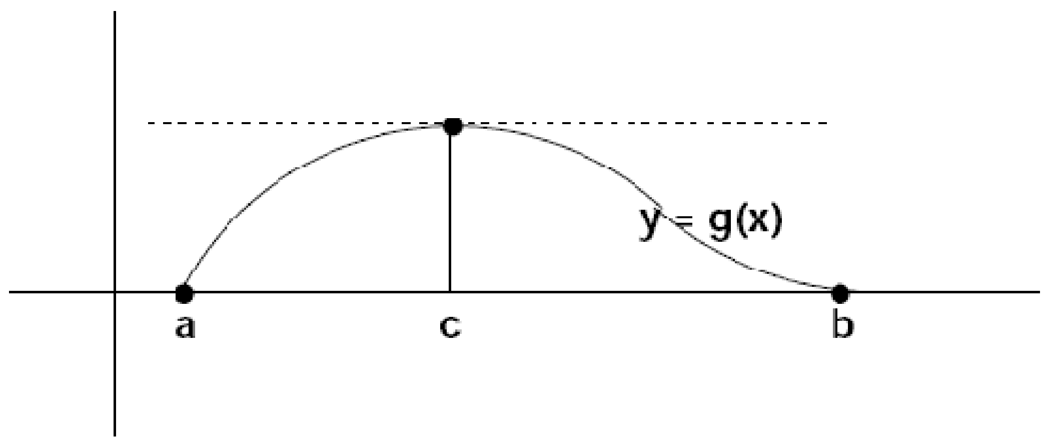
x	$f(x)$
-1	11
3	53
-4	-151
4	-39

Rolle's Theorem (general version)

Theorem. *Suppose that the function g is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). If*

$$g(a) = g(b)$$

then there exists a point c in the open interval (a, b) where $g'(c) = 0$.

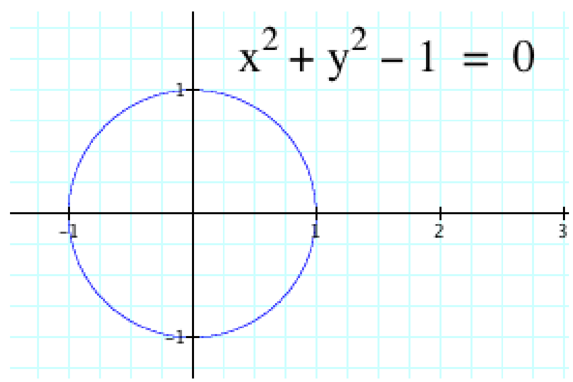


Example 7

Show that the equation $x^3 + x - 1 = 0$ has exactly one real root.

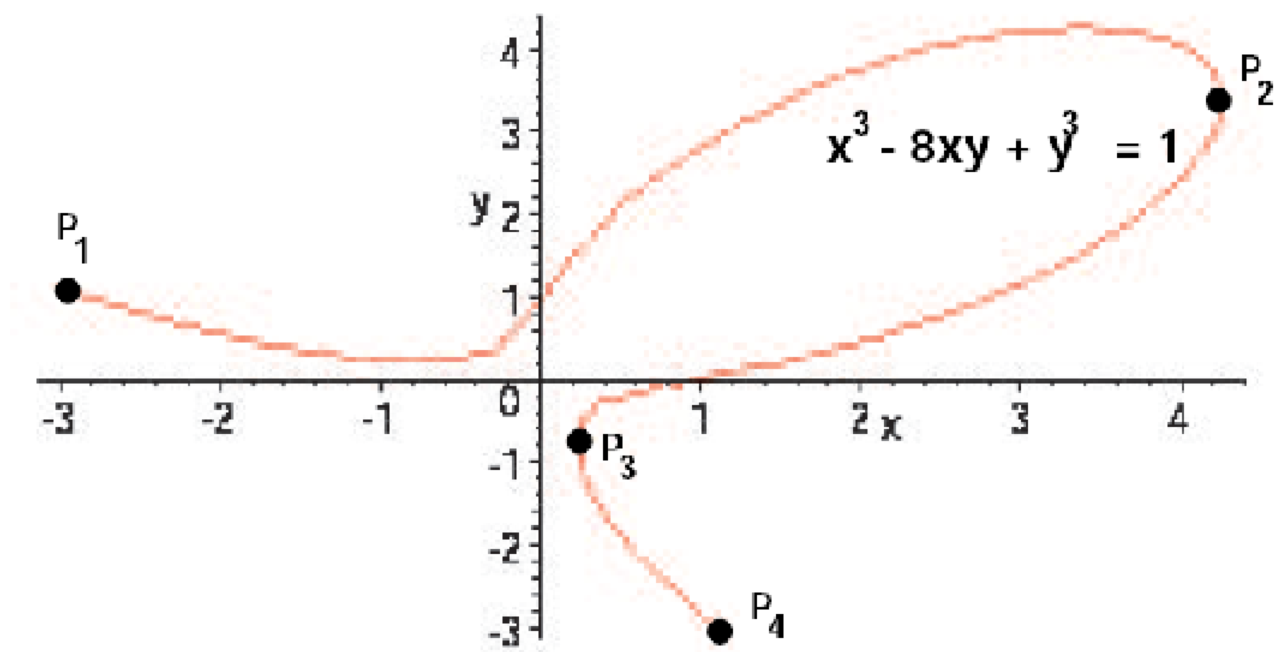
2.11: Implicit Differentiation

- Problem: Many interesting smooth curves in the plane are NOT the graphs of functions, but we still need to find tangent lines at various points (e.g., curved path of a robot in a factory.)
- A circle of radius 1, for example, does not pass the "vertical line test" and hence is not the graph of a function.
- It is, however, the graph of the equation $x^2 + y^2 - 1 = 0$.



- The equation $x^3 - 8xy + y^3 = 1$ resists our most clever efforts to explicitly solve for y as a function of x.

- We will see how to overcome this difficulty using a very important technique called implicit differentiation.



- The general setting for our discussion of implicitly defined functions is an equation $F(x, y) = 0$, where F is an expression containing the two variables x and y.

- An (unknown...) function $y = f(x)$ is implicitly defined by the equation if
$$F(x, f(x)) = 0$$
for x in some (possibly small) interval I.

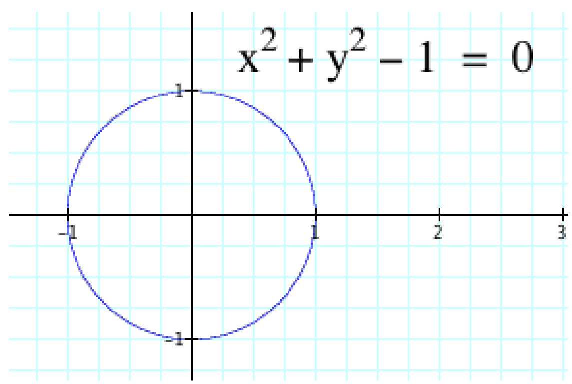
- **GOAL:** Find the derivative $\frac{dy}{dx}$ of $y = f(x)$ *without explicitly* solving the equation for y! (Is that totally cool or what!? ☺)

Example 8

- The functions $y = \sqrt{1-x^2}$ and $y = -\sqrt{1-x^2}$ are implicitly defined by the equation $x^2 + y^2 = 1$.

- Consider one of the functions $y = f(x)$ defined implicitly by the equation $x^2 + y^2 = 1$. In either case, we have that

$$f'(x) = -\frac{x}{f(x)}.$$



Example 8 (cont'd)

Given the equation $x^2 + y^2 = 1$, we think of the (unknown...) function $y = f(x)$ **implicitly** defined by the equation and differentiate it anyway using the Chain Rule!

$$x^2 + y^2 = 1$$

Example 8 (cont'd)

Given the equation $x^2 + y^2 = 1$, we think of the (unknown...) function $y = f(x)$ **implicitly** defined by the equation and differentiate it anyway using the Chain Rule!

$$\begin{aligned} x^2 + y^2 &= 1 \\ \frac{d}{dx}(x^2 + y^2) &= \frac{d}{dx}(1) \end{aligned}$$

Example 8 (cont'd)

Given the equation $x^2 + y^2 = 1$, we think of the (unknown...) function $y = f(x)$ **implicitly** defined by the equation and differentiate it anyway using the Chain Rule!

$$\begin{aligned} x^2 + y^2 &= 1 \\ \frac{d}{dx}(x^2 + y^2) &= \frac{d}{dx}(1) \\ 2x + \frac{d}{dx}(y^2) &= 0 \end{aligned}$$

Example 8 (cont'd)

Given the equation $x^2 + y^2 = 1$, we think of the (unknown...) function $y = f(x)$ **implicitly** defined by the equation and differentiate it anyway using the Chain Rule!

$$\begin{aligned} x^2 + y^2 &= 1 \\ \frac{d}{dx}(x^2 + y^2) &= \frac{d}{dx}(1) \\ 2x + \frac{d}{dx}(y^2) &= 0 \\ 2x + 2y\frac{dy}{dx} &= 0 \end{aligned}$$

Example 8 (cont'd)

Given the equation $x^2 + y^2 = 1$, we think of the (unknown...) function $y = f(x)$ **implicitly** defined by the equation and differentiate it anyway using the Chain Rule!

$$\begin{aligned} x^2 + y^2 &= 1 \\ \frac{d}{dx}(x^2 + y^2) &= \frac{d}{dx}(1) \\ 2x + \frac{d}{dx}(y^2) &= 0 \\ 2x + 2y\frac{dy}{dx} &= 0 \\ \Longrightarrow \frac{dy}{dx} &= -\frac{x}{y} \end{aligned}$$

Example 9

Use implicit differentiation to find the equation of the tangent line to the graph of the smooth curve defined implicitly by the equation

$$F(x, y) = xy^2 + x^2y - 6 = 0$$

at the point $(1, 2)$.

Example 10

Use implicit differentiation to find thederivative dy/dx of any function $y = f(x)$ implicitly defined by the equation

$$\cos x + \frac{y}{x} = \sin y + \frac{x}{y}$$

Example 10

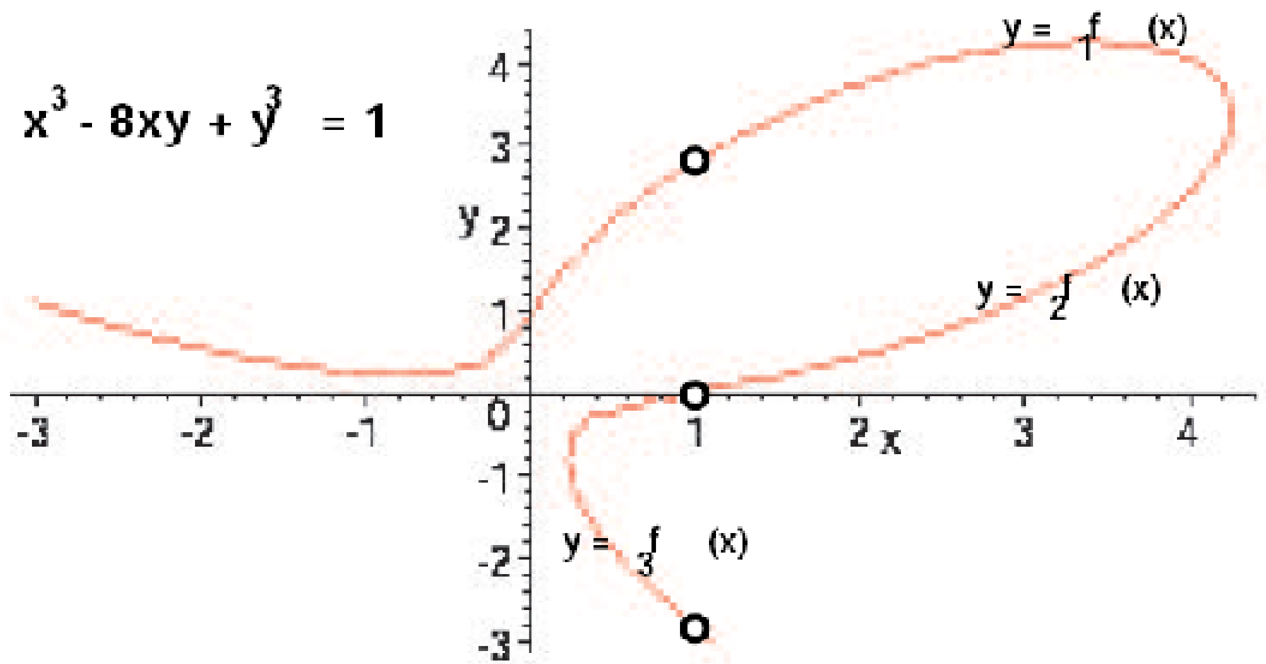
Use implicit differentiation to find the derivative dy/dx of any function $y = f(x)$ implicitly defined by the equation

$$\cos x + \frac{y}{x} = \sin y + \frac{x}{y}$$

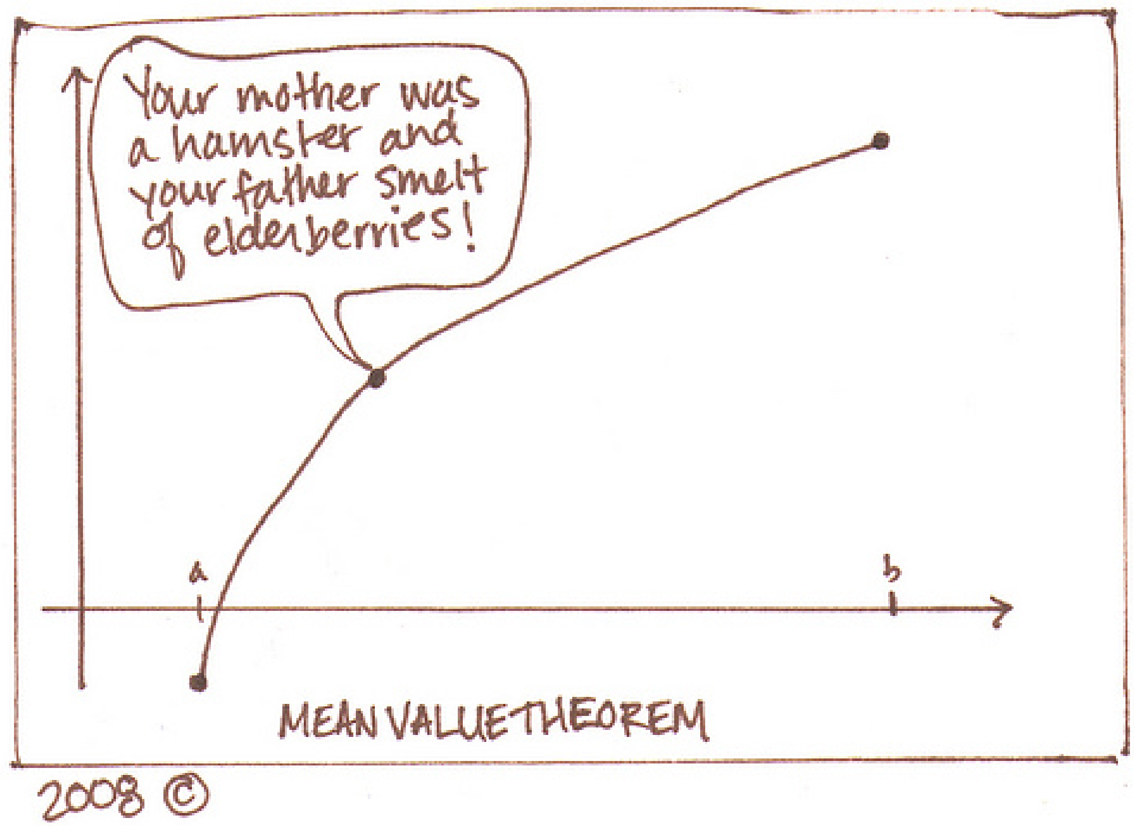
Answer: $\frac{dy}{dx} = \frac{y^3+x^2y+x^2y^2\sin(x)}{x^3+xy^2-x^2y^2\cos(y)}$

Example 11

Return to the equation $x^3 - 8xy + y^3 = 1$ with which we begin this section. Find the slope at the points on the curve for which $x = 1$.



The Mean ☹ Value Theorem by Brown Sharpie...



http://brownsharpie.courtneygibbons.org/?p=728