

# Math 43: Spring 2020

## Lecture 1 Part 1

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# What Are We Doing Here

In this course, we want to develop the calculus of complex-valued functions of a single complex variable

Let's start by thinking about what we mean by “calculus” in the first place.

Simply put, calculus begins with the study of functions

$$f : D \subset \mathbf{R} \rightarrow \mathbf{R}$$

of real-valued functions of a single real variable. Here  $D$  is the domain of  $f$ .

Then we graduate to multi-variable calculus where we work with functions

$$\overline{f}f : D \subset \mathbf{R}^n \rightarrow \mathbf{R}^m.$$

Since we're all adults here, we'll drop the bar over the  $f$ . As we shall see shortly, in this course we are interested in the case  $n = 2 = m$ . But let's not get ahead of ourselves.

# What are we going to do this term

- 1 First of all, we need to define what a complex number is.
- 2 Then we study complex arithmetic.
  - A It's more interesting than you might think.
  - B But it's still just arithmetic.
- 3 Limits and Continuity.
  - A It will turn out, that all that is required here is to apply what we learned—or where supposed to learn—in our multivariable calculus courses.
  - B My approach here is a bit different than in our text.
- 4 Complex Differentiation.
  - A This is a new concept which turns out to be much more subtle than it appears at first.
- 5 The fun bit.
  - A Complex differentiable functions are called **analytic**. We'll spend the majority of term investigating their properties.
  - B It will be helpful to recall some basics from vector calculus including line integrals and Green's Theorem.

# Why Bother?

## Remark (Do Complex Numbers Really Exist?)

We I first learned about complex numbers in high school, they were called “imaginary numbers”. But a complex number is no less real than other concepts in abstract mathematics. For example, we have all studied abstract vector spaces. These are just sets equipped with an addition and scalar multiplication satisfying certain axioms. Those of you who know a little algebra have seen other abstract concepts such as groups, rings, and fields. For that matter, have you ever given much thought to what a real number is? We happily play with them usually without knowing a formal definition.

## Remark (Hardy's Answer)

One reason to study complex numbers and their calculus is that it is simply beautiful mathematics. For example, one of the first substantive things we'll establish in this course is the formula

$$e^{j\pi} + 1 = 0.$$

This nicely ties together the 5 most important constants in mathematics. Dartmouth's own President Kemeny liked it so much that he insisted that it appear on a blackboard behind him in his presidential portrait. I put a picture of this portrait on our canvas page since we can't all truck over to the library these days.

# Another Answer

Complex numbers are there whether you like it or not!

Consider a damped spring-mass system. The motion of the suspended mass is governed by a simple ordinary differential equation:

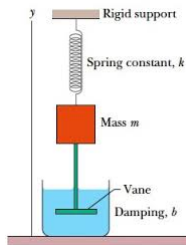
$$my'' + by' + ky = 0 \quad (1)$$

where  $y(t)$  is the displacement of the mass from equilibrium.

Since  $m, b, k > 0$ , the quadratic equation  $mr^2 + br + k = 0$  has complex roots  $r = \alpha \pm i\beta$  with  $\alpha < 0$ . The general solution to (1) is

$$y(t) = Ae^{\alpha t} \cos(\beta t) + Be^{\alpha t} \sin(\beta t)$$

where  $A$  and  $B$  are real constants determined by the initial conditions  $y(0)$  and  $y'(0)$ .



# The Cast of Characters

Let's recall that sorts of numbers we've worked with in our calculus courses up until now.

- The **natural numbers**  $\mathbf{N} = \{1, 2, 3, \dots\}$ . The choice not to include 0 in  $\mathbf{N}$  is a controversial one. Computer scientists would object.
- The **integers**, or the set of whole numbers, is the set  $\mathbf{Z} = -\mathbf{N} \cup \{0\} \cup \mathbf{N} = \{0, \pm 1, \pm 2, \dots\}$ .
- The **rational numbers**, or the set of fractions, is the set

$$\mathbf{Q} = \left\{ \frac{a}{b} : a \in \mathbf{Z} \text{ and } b \in \mathbf{N} \right\}.$$

## Remark (A Field)

The rational numbers,  $\mathbf{Q}$ , are special as they form what is called a **field**. Informally, this means we can do all the usual arithmetic operations and stay inside  $\mathbf{Q}$ . We mentioned fields above. Let's get formal.

## Definition

A **field** is a set **F** containing at least two elements 0 and 1 equipped with operations  $+$  and  $\cdot$  such that for all  $x, y, z \in \mathbf{F}$  we have  $x + y \in \mathbf{F}$  and  $x \cdot y = xy \in \mathbf{F}$  and

- |                           |   |     |  |
|---------------------------|---|-----|--|
| 1)                        | $x + y = y + x$                             | 1)' | $xy = yx$  |
| 2)                        | $x + (y + z) = (x + y) + z$                 | 2)' | $x(yz) = (xy)z$  |
| 3)                        | $x + 0 = x$                                 | 3)' | $x \cdot 1 = x$  |
| 4)                        | there exists $-x$<br>such that $-x + x = 0$ | 4)' | if $x \neq 0$ there exists $x^{-1}$<br>such that $xx^{-1} = 1$ , and |
| 5) $x(y + z) = xy + yz$ . |   |     |  |

## Example

Of course the rational numbers  $\mathbf{Q} = \left\{ \frac{a}{b} : a \in \mathbf{Z} \text{ and } b \in \mathbf{N} \right\}$  satisfy all these familiar rules of arithmetic. Hence  $\mathbf{Q}$  is a field. **But there are lots of others.**



# A Field with Four Elements

## Example

Let  $\mathbb{F}_4 = \{0, 1, a, b\}$ . Then define addition and multiplication as follows

+	0	1	a	b
0	0	1	a	b
1	1	0	b	a
a	a	b	0	1
b	b	a	1	0

·	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	b	1
b	0	b	1	a

Then it is possible to show that  $\mathbb{F}_4$  is a field. However in all honesty, it would be tedious beyond belief to check this directly. Fortunately, there are other techniques—from abstract algebra—that allow us to see this from general principles.

# Ordered Fields

As our example of a field with 4 elements shows, there is more to the “good old arithmetic” we’re used to than just the algebraic axioms of a field. We want more

## Definition

We say that a field  $\mathbf{F}$  is **ordered** if there is a subset  $P \subset \mathbf{F} \setminus \{0\}$  such that

- 1  $\mathbf{F}$  is the disjoint union of  $P$ ,  $\{0\}$ , and  $-P$ .
- 2 If  $a, b \in P$ , then  $a + b \in P$  and  $ab \in P$ .

We say that  $x > 0$  if  $x \in P$  and  $x < y$  if  $y - x \in P$ . We call the pair  $(\mathbf{F}, P)$ , or sometimes  $(F, <)$  an **ordered field**.

## Remark

If  $a$  is an element in an ordered field, either  $a$  is positive,  $-a$  is positive, or  $a = 0$ . Alternatively, given  $a, b$  in an ordered field, either  $a < b$ ,  $b < a$ , or  $a = b$ .

# Not a Rational World

## Example

Let  $P = \{ \frac{a}{b} \in \mathbf{Q} : a, b \in \mathbf{N} \}$ . Then  $(\mathbf{Q}, P)$  is an ordered field that we've held dear to our hearts since grade school.

- The rationals were fine for middle school, but we quickly see that there are numbers like  $\sqrt{3}$ ,  $e$ , and  $\pi$  that are not rational.
- Thus if we want to model the real world, we need to enlarge  $\mathbf{Q}$  so that, for example, every cubic polynomial crosses the  $x$ -axis (that is, has a root). We also want to write down a formula for the area of a circle of radius one, or to describe exponential growth, and generally perform many other tasks that take us outside of the “rational” world.
- Since we still want to do arithmetic using the usual axioms that make  $\mathbf{Q}$  a field, we want a **field**  $\mathbf{R}$  that contains not only all the fractions in  $\mathbf{Q}$  but **all** the other “numbers” we need to model the world we live in. That is, the “real” world.

# The Real Numbers

## Remark

Without much fanfare in High school, it was asserted that there was a field  $\mathbf{R}$ —called the field of **Real numbers**—such that  $\mathbf{Q} \subset \mathbf{R}$  and “everything we wanted” was in  $\mathbf{R}$ . Formally, we want  $\mathbf{R}$  to be a complete ordered field in that it satisfies the following property:

Given a non-empty set  $S \subset \mathbf{R}$  such that there is a  $b \in \mathbf{R}$  such that  $s \leq b$  for all  $s \in S$ , then there is a  $u \in \mathbf{R}$  such that

- ①  $s \leq u$  for all  $s \in S$ , and
- ② if  $s \leq t$  for all  $s \in S$ , then  $u \leq t$ .

Then  $u$  is called the **least upper bound** of  $S$ . We write  $u = \text{lub}(S)$ .

# Complete is the Thing

## Example

Let

$$S = \{r \in \mathbf{Q} : r^2 < 3\}.$$

Then  $S$  is bounded above and  $\sqrt{3} = \text{lub}(S)$ . The real point of this example—no pun intended—is that in the field  $\mathbf{Q}$ , the set  $S$  is still bounded above, but it has no least upper bound.

## Remark

Remarkably, the completeness axiom tells us that  $\mathbf{R}$  has to contain just about everything we want. Just as in all the courses that have come before, we just assume that the real numbers exist and form the play ground we are used to. Of course, depending on your darkest desires,  $\mathbf{R}$  may actually not contain “everything you want”. We know that there is no real number  $x$  such that  $x^2 = -1$ .

# Enough for one Go



With all due apologies to Scott Adams, perhaps it is time to take a break.