# Math 43: Spring 2020
# Lecture 1 Part 2

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It is time we gave a formal definition of what we mean by a complex number.

**Definition**

We let $\mathbf{C}$ be the set $\mathbf{R}^2 = \{ (x,y) : x,y \in \mathbf{R} \}$ together with the operations of addition given by

$$(x,y) + (x',y') = (x+x', y+y')$$

and multiplication given by

$$(x,y)(x',y') = (xx' - yy', xy' + x'y).$$

We call $\mathbf{C}$ the set of complex numbers.

# Say What?

**Remark**

The first thing to observe, is that the set of real numbers, $\mathbf{R}$, sits inside $\mathbf{C}$ in a natural way: $x \in \mathbf{R} \mapsto (x, 0) \in \mathbf{C}$. Furthermore $x + y \mapsto (x, 0) + (y, 0)$ while $xy \mapsto (x, 0)(y, 0)$.

One other cool thing is that

$$(0, 1)(0, 1) = (0, 1)^2 = (-1, 0).$$

Therefore if we define $i = (0, 1)$, then we've found a square root of $-1$; that is, $i^2 = -1$. Futhermore, if we identify $x \in \mathbf{R}$ with $(x, 0)$ in $\mathbf{C}$, then

$$(x, y) = (x, 0) + (0, y) = (x, 0) + (0, 1)(y, 0) \text{ ``$=$'' } x + iy.$$

**Remark**

In view of the previous slide, we will almost always write $(x, y) \in \mathbf{C}$ as the sum $x + iy$. But when push comes to shove, you best remember $x + iy$ is just a short hand for the ordered pair $(x, y)$. One advantage of this notation—besides the fact that that is the notation you know from high school—is that it explains the formulas for addition and multiplication in the definition of $\mathbf{C}$.

$$\begin{aligned}(x+iy)+(x'+iy') &= x+x'+i(y+y') \quad \text{and} \\ (x+iy)(x'+iy') &= xx'+x(iy')+iy(x')+iy(iy') \\ &= xx'-yy'+i(xy'+x'y).\end{aligned}$$

# $\mathbf{C}$ is a Field

**Theorem**

*The complex numbers* $\mathbf{C}$ *form a field with zero element* $0 = (0,0) = 0 + i0$ *and multiplicative identity* $1 = (1,0) = 1 + i0$.

**Proof.**

This can be proved by boring calculations using the fact that $\mathbf{R}$ is a field. The one tricky bit is the existence of multiplicative inverses. That is we have to verify the axiom that says given $a + ib \neq 0$, then there is an element $(a+ib)^{-1} = x + iy$ such that $1 = (a+ib)(x+iy) = (ax - by) + i(ay + bx)$. Thus we need to solve

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} ax - by \\ bx + ay \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

for $x$ and $y$. You can check that the solution is $x = \frac{a}{a^2+b^2}$ and $y = \frac{-b}{a^2+b^2}$. Thus

$$(a+ib)^{-1} = \frac{a}{a^2+b^2} + i\frac{-b}{a^2+b^2}.$$

$\square$

# Basics

**Theorem (Homework)**

*The complex numbers can not be ordered. That is, unlike* $\mathbf{Q}$ *and* $\mathbf{R}$, *the set* $\mathbf{C}$ *of complex numbers is not an ordered field.*

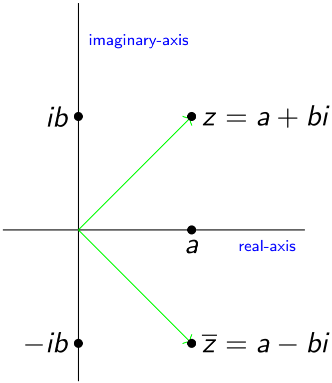
**Definition**

If $z = x + iy \in \mathbf{C}$, then we define the real part of $z$ to be $\mathrm{Re}(z) = x$ and the imaginary part of $z$ to be $\mathrm{Im}(z) = y$. The complex conjugate of $z$ is $\overline{z} = x - iy$.

**Remark**

It is worth pointing out that both $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ are real numbers. Do not make the mistake of thinking that $\mathrm{Im}(x + iy)$ should be $iy$.

We
must always keep in mind that $\mathbf{C}$ has (no pun intended) a complex geometry coming from $\mathbf{R}^2$.
Thus we can think of $z = a + ib$ as the vector $\langle a, b \rangle$ in $\mathbf{R}^2$.
If $z = a + ib \in \mathbf{C}$,
then we define the modulus of $z$ to be $|z| = \sqrt{a^2 + b^2}$.
Note that in the vector picture, $|z|$ is just the length $\|\langle a, b \rangle\|$ of the vector $\langle a, b \rangle$. This is a critical observation for drawing pictures.



# Playing with the Complex Conjugate

One special property of the complex conjugate it the following: suppose $z = a + ib$. Then

$$z\overline{z} = (a + ib)(a - ib) = a^2 - iab + iab + b^2 = a^2 + b^2 = |z|^2.$$

The following other identities are proved similarly.

**Lemma**

*Let $z, w \in \mathbf{C}$.*

1. $z\overline{z} = |z|^2$.
2. $\overline{z^{-1}} = \overline{z}^{-1}$.
3. $\overline{zw} = \overline{z}\,\overline{w}$.
4. $\overline{z/w} = \overline{z}/\overline{w}$. *(Hint: $z/w := zw^{-1}$!)*
5. $\overline{\overline{z}} = z$.
6. $\frac{1}{z} = \frac{\overline{z}}{|z|^2}$

# Comment on Division

## Remark

The last formula on the previous slide deserves our attention:

$$\frac{1}{z} = \frac{1}{z} \cdot \frac{\overline{z}}{\overline{z}} = \frac{\overline{z}}{|z|^2}.$$

As is often the case with mathematical proofs, they show us how to do things in the future. Consider

$$\begin{aligned} \frac{1+i}{3-2i} = \frac{1+i}{3-2i}\Big(\frac{3+2i}{3+2i}\Big) &= \frac{(1+i)(3+2i)}{3^2+2^2} \\ = \frac{3-2+5i}{13} &= \frac{1}{13} + i\frac{5}{13}. \end{aligned}$$

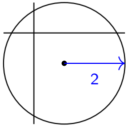
# Don't Forget the Geometry!

**Example**

Describe the set $A = \{ z \in \mathbf{C} : |z - 1 + i| = 2 \}$.

**Solution.**

Remember $|z - w|$ is the length of the corresponding vector! Hence $A$ is the set of points $z$ whose distance to $1 - i$ is 2. Thus $A$ is the circle in the plane of radius 2 centered at $1 - i$ or $(1, -1)$ if you prefer. Alternatively, note that
$A = \{ z = x + iy : \sqrt{(x-1)^2 + (y+1)^2} = 2 \}$. □



# A Challenge

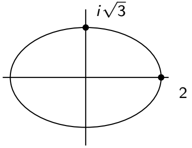
**Example**

Describe the set $B = \{\, z \in \mathbf{C} : |z-1| + |z+1| = 4 \,\}$.

On the one hand, $B$ is the set of points $z$ such that the sum of the distance from $z$ to 1 and from $z$ to $-1$ is 4. If you had a good high school geometry class, you know that this is an ellipse with foci 1 and $-1$.

With a lot more algebra, you can use $z\bar{z} = |z|^2$ and complex arithmetic to verify that

$$B = \{\, z = x + iy : \frac{x^2}{4} + \frac{y^2}{3} = 1 \,\}.$$




I leave checking that last equation as a challenge.