Math 43: Spring 2020
Lecture 10 Summary

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Contours

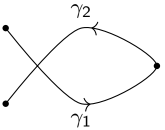
Definition

A contour Γ in $\mathbf{C}$ consists either of a single point z_0, or a finite sequence $\gamma_1, \dots, \gamma_n$ of directed smooth curves such that the terminal point of γ_k is the starting point of γ_{k+1} for $1 \leq k < n$. In the case Γ is not a single point, we write $\Gamma = \gamma_1 + \cdots + \gamma_n$.

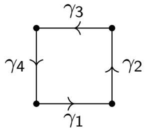
Definition

An admissible parameterization of a contour $\Gamma = \gamma_1 + \cdots + \gamma_n$ is obtained by concatinating admissible pasteurization's of the γ_k. Thus given a partition $a = \tau_0 < \tau_1 < \cdots < \tau_n = b$ of $[a, b]$, then we require admissible parameterizations $z_k : [\tau_{k-1}, \tau_k] \to \mathbf{C}$ of γ_k. Then we obtain an admissible parameterization $z : [a, b] \to \mathbf{C}$ for Γ by letting $z(t) = z_k(t)$ if $t \in [\tau_{k-1}, \tau_k]$. If Γ is a single point, then we allow a constant function as an admissible parameterization.

Examples



(a) $\Gamma = \gamma_1 + \gamma_2$



(b) $\Gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$

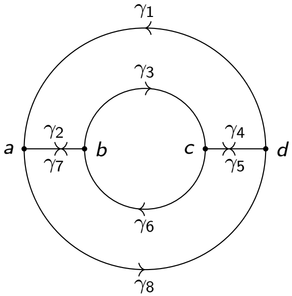


Figure: $\Gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_8$ with $\gamma_7 = -\gamma_2$ and $\gamma_5 = -\gamma_4$

Closed Contours

Definition

A Contour $\Gamma = \gamma_1 + \cdots + \gamma_n$ is closed if the terminal point of γ_n is the initial point of γ_1. We will sometimes use the term loop to speak of a closed contour. We call a closed contour simple if it intersects itself only at its endpoints.

Remark

If $z : [a, b] \to \mathbf{C}$ is an admissible parameterization of a contour Γ, then Γ is simple closed contour if and only if z is one-to-one on $[a, b)$ and $z(a) = z(b)$. Note that closed smooth curve is a simple closed contour.

The Jordan Curve Theorem

Theorem (The Jordan Curve Theorem)

A simple closed contour Γ separates the plane into two domains each having Γ as their boundary. One of these domains is bounded and is called the interior of Γ. the other is called the exterior of Γ.

Remark

We will accept the Jordan Curve Theorem as "clearly true". However, it is not very easy to give a proof. Perhaps the first observation to be made is that it even needs a proof. As it turns out, Jordan Curves—that is simple closed paths—can be complex beyond belief. As we saw by example in lecture, even Jordan contours can be pretty complex.

Positive Orientation

Definition

We say that a simple closed contour is positively oriented if the interior is on your left as you transverse the curve.

Example

The usual parametrization $z(t) = z_0 + re^{it}$ with $t \in [0, 2\pi]$ is a positively oriented circle of radius r centered at z_0.

Arc Length

Definition

If γ is a smooth curve with admissible parameterization $z : [a, b] \to \mathbf{C}$ given by $z(t) = x(t) + iy(t)$, then the length of γ is given by

$$\ell(\gamma) = \int_a^b |z'(t)|\,dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,dt.$$

If $\Gamma = \gamma_1 + \cdots + \gamma_n$, then $\ell(\Gamma) = \sum_{k=1}^n \ell(\gamma_k)$.

Remark

We know from multivariable calculus that $\ell(\gamma)$, and hence $\ell(\Gamma)$, is independent of admissible parameterization.

Ordinary Inegrals

Remark

The approach in the text to integrals of complex-valued functions of a real variable is, to my mind, unnecessarily complicated. We will take a simplified approach.

Definition

If $z(t) = u(t) + iv(t)$ and $z : [a, b] \to \mathbf{C}$ is continuous, then we define

$$\int_a^b z(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt.$$

The Fundamental Theorem is still Fundamental

Lemma

Suppose that $z(t) = u(t) + iv(t)$ *is continuous on* $[a, b]$ *and that* $F : [a, b] \to \mathbf{C}$ *is such that* $F'(t) = z(t)$*. Then*

$$\int_a^b z(t)\,dt = F(t)\Big|_a^b = F(b) - F(a).$$

Example

$$\int_0^{\frac{\pi}{2}} e^{2it}\,dt = \frac{e^{2it}}{2i}\,\Big|_0^{\frac{\pi}{2}} = \frac{1}{2i}(e^{i\pi} - e^0) = \frac{-1}{i} = i.$$

Contour Integrals

Definition

Let γ be a directed smooth curve with admissible parameterization $z : [a, b] \to \mathbf{C}$. If f is a complex-valued function which is continuous on γ, then we define the contour integral of f over γ to be

$$\int_\gamma f(z)\,dz = \int_a^b f\big(z(t)\big)z'(t)\,dt.$$

Theorem

Let C_r be the positively oriented circle of radius r centered at z_0. If $n \in \mathbf{Z}$, then

$$\int_{C_r} (z - z_0)^n\,dz = \begin{cases} 2\pi i & \text{if } n = -1, \text{ and} \\ 0 & \text{if } n \neq -1. \end{cases}$$

Extending to Arbitrary Contours

Definition

Let Γ be a contour. Suppose f is continuous on Γ. If $\Gamma = \gamma_1 + \cdots + \gamma_n$, then we define the contour integral of f over Γ to be

$$\int_\Gamma f(z)\, dz = \sum_{k=1}^{n} \int_{\gamma_k} f(z)\, dz.$$

If Γ consists of a single point, then we define the contour integral to be zero.

Remark

If Γ is the single point z_0, we will consider the constant function $z(t) = z_0$ for all $t \in [a, b]$ to be an admissible parameterization of Γ. Then

$$\int_\Gamma f(z)\, dz = \int_a^b f(z(t)) z'(t)\, dt = 0$$

since $z'(t) \equiv 0$.