

Math 43: Spring 2020

Lecture 10 Part 1

Dana P. Williams

Dartmouth College

Monday April 20, 2020

Remark

We talked at length about smooth curves last time. This is really old material from multivariable calculus, but you would be wise to review our terminology and the basic definitions before watching this lecture. As it turns out, we won't want to restrict our attention exclusively to smooth curves. We require slightly more general paths.

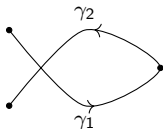
Definition

A **contour** Γ in \mathbf{C} consists either of a single point z_0 , or a finite sequence $\gamma_1, \dots, \gamma_n$ of **directed** smooth curves such that the terminal point of γ_k is the starting point of γ_{k+1} for $1 \leq k < n$. In the case Γ is not a single point, we write $\Gamma = \gamma_1 + \dots + \gamma_n$.

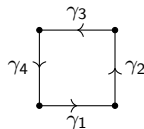
Remark

Of course, any smooth curve is a contour. But there are lots of reasonable paths—besides single points—that aren't smooth curves.

Examples



(a) $\Gamma = \gamma_1 + \gamma_2$



(b) $\Gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$

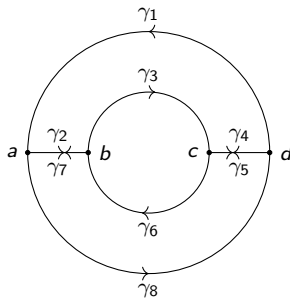


Figure: $\Gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_8$ with $\gamma_7 = -\gamma_2$ and $\gamma_5 = -\gamma_4$

Parameterizing Contours

Definition

An admissible parameterization of a contour $\Gamma = \gamma_1 + \cdots + \gamma_n$ is obtained by concatenating admissible pasteurizations of the γ_k . Thus given a partition $a = \tau_0 < \tau_1 < \cdots < \tau_n = b$ of $[a, b]$, then we require admissible parameterizations $z_k : [\tau_{k-1}, \tau_k] \rightarrow \mathbf{C}$ of γ_k . Then we obtain an admissible parameterization $z : [a, b] \rightarrow \mathbf{C}$ for Γ by letting $z(t) = z_k(t)$ if $t \in [\tau_{k-1}, \tau_k]$.

Remark

This isn't as bad as it sounds. It will usually be enough to realize that a contour has an admissible parameterization and we won't have to write one down. But when we do, the following trick is useful. If $z : [a, b] \rightarrow \mathbf{C}$ is an admissible parameterization of γ then so is $w : [a + c, b + c] \rightarrow \mathbf{C}$ where $w(t) = z(t - c)$.

An Example

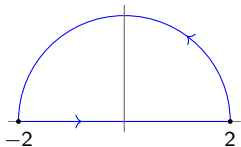
Example

Let $\Gamma = \gamma_1 + \gamma_2$ be the contour consisting of $\gamma_1 = [-2, 2]$ and γ_2 is the top half of the circle $|z| = 2$ from 2 to -2 . Find an admissible parameterization.

Solution:

We can parameterize γ_1 by $z_1(t) = t$ with $t \in [-2, 2]$. We can parameterize γ_2 by $w(t) = 2e^{it}$ with $t \in [0, \pi]$. (We only want to go halfway around the circle.) We use the trick and let $z_2 : [2, 2 + \pi]$ be given by $z_2(t) = w(t - 2) = 2e^{i(t-2)}$. Then we get our admissible parameterization $z : [0, 2 + \pi] \rightarrow \mathbf{C}$ by

$$z(t) = \begin{cases} z_1(t) & \text{if } t \in [-2, 2] \text{ and} \\ z_2(t) & \text{if } t \in [2, \pi + 2]. \end{cases}$$



Closed Contours

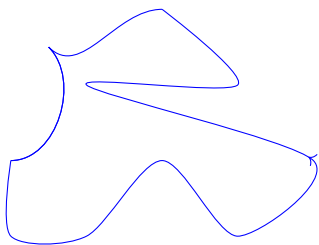
Definition

A Contour $\Gamma = \gamma_1 + \cdots + \gamma_n$ is **closed** if the terminal point of γ_n is the initial point of γ_1 . We will sometimes use the term **loop** to speak of a closed contour. We call a closed contour **simple** if it intersects itself only at its endpoints.

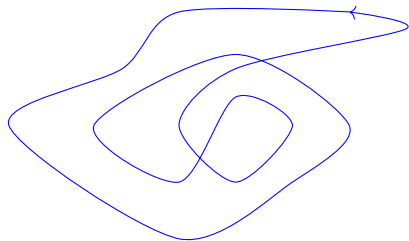
Remark

If $z : [a, b] \rightarrow \mathbf{C}$ is an admissible parameterization of a contour Γ , then Γ is simple closed contour if and only if z is one-to-one on $[a, b)$ and $z(a) = z(b)$. Note that closed smooth curve is a simple closed contour.

Examples of Closed Contours



(a) A Simple Closed Contour



(b) A Smooth Non-Simple Closed Contour

Figure: Closed Contours

The Jordan Curve Theorem

Theorem (The Jordan Curve Theorem)

A simple closed contour Γ separates the plane into two domains each having Γ as their boundary. One of these domains is bounded and is called the interior of Γ . the other is called the exterior of Γ .

Remark

We will accept the Jordan Curve Theorem as “clearly true”. However, it is not very easy to give a proof. Perhaps the first observation to be made is that it even needs a proof. As it turns out, Jordan Curves—that is simple closed paths—can be complex beyond belief. As the figure on the next slide shows, even Jordan contours can be pretty complex.

A Jordan Contour

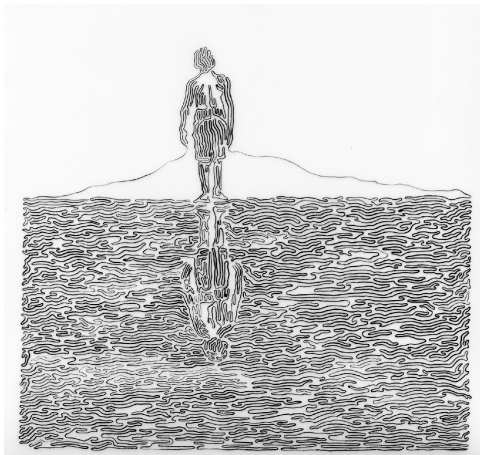


Figure: "A Thread in the Labyrinth" by Fiona Ross (2011). Can you spot the interior of this curve?

Positive Orientation

Definition

We say that a simple closed contour is **positively oriented** if the interior is on your left as you transverse the curve.

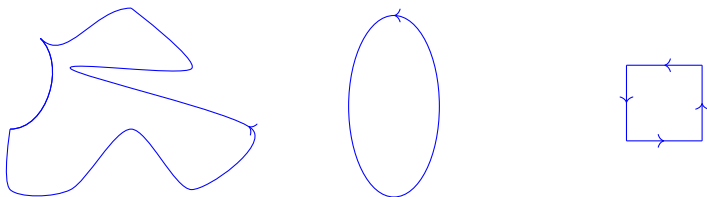


Figure: Positively Oriented Simple Closed Contours

Example

The usual parametrization $z(t) = z_0 + re^{it}$ with $t \in [0, 2\pi]$ is a positively oriented circle of radius r centered at z_0 .

Arc Length

Definition

If γ is a smooth curve with admissible parameterization $z : [a, b] \rightarrow \mathbf{C}$ given by $z(t) = x(t) + iy(t)$, then the length of γ is given by

$$\ell(\gamma) = \int_a^b |z'(t)| dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2} dt.$$

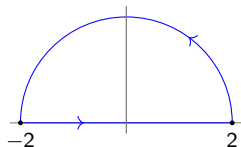
If $\Gamma = \gamma_1 + \cdots + \gamma_n$, then $\ell(\Gamma) = \sum_{k=1}^n \ell(\gamma_k)$.

Remark

We know from multivariable calculus that $\ell(\gamma)$, and hence $\ell(\Gamma)$, is independent of admissible parameterization.

An Example

Let $\Gamma = \gamma_1 + \gamma_2$ be the contour consisting of $\gamma_1 = [-2, 2]$ and γ_2 is the top half of the circle $|z| = 2$ from 2 to -2 . Then $\ell(\Gamma) = \ell(\gamma_1) + \ell(\gamma_2) = 4 + 2\pi$.



Enough. Time for a Break