Math 43: Spring 2020
Lecture 10 Part 2

Dana P. Williams

Dartmouth College

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Remark

There is no real two-dimensional analogue for complex integration. After all, we showed on homework that not all analytic functions have antiderivatives—we showed there is no antiderivative of $f(z) = \frac{1}{z}$ in the punctured complex plane (§3.3, #14). As a result, we work only with complex valued functions of a real variable. The approach in the text is to my mind, unnecessarily complicated. We will take a simplified approach.

Definition

If $z(t) = u(t) + iv(t)$ and $z : [a, b] \to \mathbf{C}$ is continuous, then we define

$$\int_a^b z(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt.$$

The Fundamental Theorem is still Fundamental

Lemma

Suppose that $z(t) = u(t) + iv(t)$ *is continuous on* $[a, b]$ *and that* $F : [a, b] \to \mathbf{C}$ *is such that* $F'(t) = z(t)$. *Then*

$$\int_a^b z(t)\,dt = F(t)\Big|_a^b = F(b) - F(a).$$

Proof.

Let $F(t) = U(t) + iV(t)$. Then by assumption $U'(t) = u(t)$ and $V'(t) = v(t)$. Then by the usual Fundamental Theorem of Calculus,

$$\begin{aligned} \int_a^b z(t)\,dt &= \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt \\ &= U(t)\Big|_a^b + iV(t)\Big|_a^b = F(t)\Big|_a^b \\ &= F(b) - F(a). \end{aligned}$$

□

Basic Example

Lemma

Suppose that $a \in \mathbf{R}$ and $w(t) = e^{iat}$. Then $w'(t) = aie^{iat}$.

Proof.

We have $w(t) = \cos(at) + i\sin(at)$. Hence $w'(t) = -a\sin(at) + ia\cos(at) = ia(i\sin(at) + \cos(at)) = iaw(t)$. $\square$

Example

$$\int_0^{\frac{\pi}{2}} e^{2it}\,dt = \frac{e^{2it}}{2i}\Big|_0^{\frac{\pi}{2}} = \frac{1}{2i}(e^{i\pi} - e^0) = \frac{-1}{i} = i.$$

A Little Calculus From Back in the Day

Lemma

Suppose that $z : [a, b] \to \mathbf{C}$ *and* $\varphi : [c, d] \to [a, b]$ *is are differentiable. Let* $w(s) = z(\varphi(s))$*. Then* $w'(s) = z'(\varphi(s))\varphi'(s)$*.*

Proof.

Let $z(t) = x(t) + iy(t)$. Then $w(s) = x(\varphi(s)) + iy(\varphi(s))$. Hence $w'(s) = x'(\varphi(s))\varphi'(s) + iy'(\varphi(s))\varphi'(s)$. Since $\varphi'(s)$ is real, $w'(s) = z'(\varphi(s))\varphi'(s)$. □

Lemma

Let γ *be a directed smooth curve. Suppose that* $z : [a, b] \to \mathbf{C}$ *and* $w : [c, d] \to \mathbf{C}$ *are both admissible parameterizations of* γ*. If* f *is a continuous complex-valued function on* γ*, then*

$$\int_a^b f(z(t))z'(t)\,dt = \int_c^d f(w(t))w'(t)\,dt.$$

The Proof

Proof.

For simplicity, assume γ is a directed smooth arc. Then z and w are both one-to-one and onto with $z(a) = w(c)$ and $z(b) = w(d)$. Then we can define $\varphi : [c, d] \to [a, b]$ by $\varphi(s) = z^{-1}(w(s))$. Some not so trivial calculus implies that φ is differentiable. Since $w(s) = z(\varphi(s))$, we have $w'(s) = z'(\varphi(s))\varphi'(s)$ and

$$\int_c^d f(w(s))w'(s)\,ds = \int_c^d f(z(\varphi(s)))z'(\varphi(s))\varphi'(s)\,ds$$

which, after $t = \varphi(s)$ and $dt = \varphi'(s)\,ds$, is

$$\begin{aligned} &= \int_{\varphi(c)}^{\varphi(d)} f(z(t))z'(t)\,dt \\ &= \int_a^b f(z(t))z'(t)\,dt. \end{aligned}$$

□

Definition

Let γ be a directed smooth curve with admissible parameterization $z : [a, b] \to \mathbf{C}$. If f is a complex-valued function which is continuous on γ, then we define the contour integral of f over γ to be

$$\int_\gamma f(z)\,dz = \int_a^b f\big(z(t)\big)z'(t)\,dt.$$

Remark

1. The whole point of the previous technical foray into calculus is that the definition of $\int_\gamma f(z)\,dz$ is independent of the admissible parameterization choosen!
2. The text avoids this by making the definition a theorem. But I have choosen what I hope is a simpler approach.

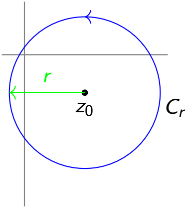
A Fundamental Example

Theorem

Let C_r be the positively oriented circle of radius r centered at z_0. If $n \in \mathbf{Z}$, then

$$\int_{C_r} (z - z_0)^n \, dz = \begin{cases} 2\pi i & \textit{if } n = -1, \textit{ and} \\ 0 & \textit{if } n \neq -1. \end{cases}$$

Here $f(z) = (z - z_0)^n$.



The Proof

Proof.

We can parameterize C_r by $z(t) = z_0 + re^{it}$ for $t \in [0, 2\pi]$. Then by definition

$$\begin{aligned}
\int_{C_r} f(z)\,dz &= \int_0^{2\pi} f(z(t))z'(t)\,dt \\
&= \int_0^{2\pi} \bigl(z(t) - z_0\bigr)^n z'(t)\,dt \\
&= \int_0^{2\pi} (re^{it})^n rie^{it}\,dt \\
&= ir^{n+1} \int_0^{2\pi} e^{i(n+1)}\,dt
\end{aligned}$$

Proof Continued

Proof Continued.

If $n \neq -1$, then

$$ir^{n+1}\int_0^{2\pi} e^{i(n+1)t}\,dt = ir^{n+1}\Big(\frac{e^{i(n+1)t}}{i(n+1)}\Big)\Big|_0^{2\pi} = \frac{r^{n+1}}{n+1}(1-1) = 0.$$

But if $n = -1$, then

$$ir^{n+1}\int_0^{2\pi} e^{i(n+1)}\,dt = i\int_0^{2\pi} 1\,dt = 2\pi i. \qquad \square$$

Extending to Arbitrary Contours

Definition

Let Γ be a contour. Suppose f is continuous on Γ. If $\Gamma = \gamma_1 + \cdots + \gamma_n$, then we define the contour integral of f over Γ to be

$$\int_\Gamma f(z)\,dz = \sum_{k=1}^{n} \int_{\gamma_k} f(z)\,dz.$$

If Γ consists of a single point, then we define the contour integral to be zero.

Remark

If Γ is the single point z_0, we will consider the constant function $z(t) = z_0$ for all $t \in [a, b]$ to be an admissible parameterization of Γ. Then

$$\int_\Gamma f(z)\,dz = \int_a^b f(z(t))z'(t)\,dt = 0$$

since $z'(t) \equiv 0$.

An Example

Let $\Gamma = [-2,2] + C_2^+$ where C_2^+ is the top half of the circle $|z| = 2$ from 2 to -2. We want to evaluate $I := \int_\Gamma (\overline{z})^2\,dz$. Fortunately, we don't have to bother parameterizing Γ! We have $I = \int_{[-2,2]} (\overline{z})^2\,dz + \int_{C_r^+} (\overline{z})^2\,dz$.

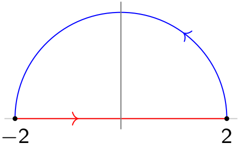


Figure: $\Gamma = [-2,2] + C_2^+$

For $[-2,2]$ we can take $z_1(t) = t$ for $t \in [-2,2]$. Then $\int_{[-2,2]} (\overline{z})^2\,dz = \int_{-2}^{2} t^2\,dt = \dfrac{16}{3}$. For C_2^+, we can let $z_2(t) = 2e^{it}$ for $t \in [0,\pi]$. Then $\int_{C_2^+} (\overline{z})^2\,dz = \int_0^\pi (2e^{-it})^2 2ie^{it}\,dt =$

$8i \int_0^\pi e^{-it}\,dt = 8i\left(\dfrac{1}{-i}\right)e^{-it}\Big|_0^\pi = -8(e^{-i\pi} - e^0) = 16$. Thus $\boxed{I = \frac{16}{3} + 16}$.

Enough

That is enough for Today.