Math 43: Spring 2020
Lecture 11 Summary

Dana P. Williams

Dartmouth College

Wednesday April 22, 2020

Back to Integrals

Lemma

Suppose that $w : [a, b] \to \mathbf{C}$ *is continuous.*

1. *If* $\alpha \in \mathbf{C}$*, then* $\alpha \int_a^b w(t)\,dt = \int_a^b \alpha w(t)\,dt$.
2. $\left|\int_a^b w(t)\,dt\right| \le \int_a^b |w(t)|\,dt$.

Standard Properties

Theorem

Suppose that f and g are continuous on a contour Γ.

1. $\displaystyle\int_\Gamma (f(z) + g(z))\,dz = \int_\Gamma f(z)\,dz + \int_\Gamma g(z)\,dz.$
2. *If* $\alpha \in \mathbf{C}$, *then* $\displaystyle\alpha \int_\Gamma f(z)\,dz = \int_\Gamma \alpha f(z)\,dz.$
3. *If* $-\Gamma$ *is the opposite contour, then* $\displaystyle\int_{-\Gamma} f(z)\,dz = -\int_\Gamma f(z)\,dz.$
4. *If* $|f(z)| \le M$ *for all* $z \in \Gamma$, *then*

$$\Big|\int_\Gamma f(z)\,dz\Big| \le M\ell(\Gamma).$$

An Example

Example

Let C_2 be the positively oriented circle $|z| = 2$. Show that

$$\Big|\int_{C_2} \frac{\cos(z)}{z^3+1}\,dz\Big| \leq \frac{4}{7}e^2\pi$$

Fundamental Theorem for Contour Integrals

Theorem (Fundamental Theorem for Contour Integrals)

Suppose that f is continuous on a domain D and that F is an antiderivative for f in D. (That is, $F'(z) = f(z)$ for all $z \in D$.) If Γ is a contour in D from w_1 to w_2, then

$$\int_\Gamma f(z)\,dz = F(w_2) - F(w_1).$$

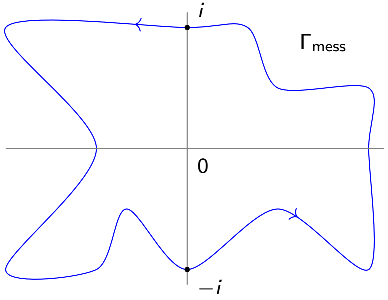
Some Fun

Consider
the complicated contour
Γ_{mess} drawn at right. Then

$$\int_{\Gamma_{\text{mess}}} e^{2z}\,dz = \frac{e^{2z}}{2}\Big|_{i}^{i} = 0.$$

Simple. But what about

$$\int_{\Gamma_{\text{mess}}} \frac{1}{z}\,dz?$$



We know that there can be no
antiderivative for $\frac{1}{z}$ in a domain that contains the contour Γ_{mess}!
But we can work a bit harder. We verified that by breaking up Γ_{mess}
into pieces where the FTCI applies, that $\displaystyle\int_{\Gamma_{\text{mess}}} \frac{1}{z}\,dz = 2\pi i$.