Math 43: Spring 2020
Lecture 11 Part 1

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Remark

Now we want to explore some basic properties of integrals of complex-valued functions of a real variable and especially of contour integrals. It might be wise to review the last part of the previous lecture before proceeding.

Lemma

Suppose that $w : [a, b] \to \mathbf{C}$ is continuous.

1. *If $\alpha \in \mathbf{C}$, then* $\alpha \int_a^b w(t)\,dt = \int_a^b \alpha w(t)\,dt$.
2. $\left| \int_a^b w(t)\,dt \right| \leq \int_a^b |w(t)|\,dt$.

Proof of (1)

Proof.

Let $w(t) = u(t) + iv(t)$ and $\alpha = x + iy$. Then

$$\alpha \int_a^b w(t)\,dt = (x+iy)\Big(\int_a^b u(t)\,dt + i\int_a^b v(t)\,dt\Big) =$$

$$x\int_a^b u(t)\,dt - y\int_a^b v(t)\,dt + i\Big(y\int_a^b u(t)\,dt + x\int_a^b v(t)\,dt\Big) =$$

$$\int_a^b \big(xu(t) - yv(t)\big)\,dt + i\int_a^b \big(yu(t) + xv(t)\big)\,dt\Big) =$$

$$\int_a^b (x+iy)(u(t)+iv(t))\,dt = \int_a^b \alpha w(t)\,dt.$$

□

Proof of (2)

Proof.

Let $\int_a^b w(t)\,dt = re^{i\theta}$. Then

$$\begin{aligned}\Bigl|\int_a^b w(t)\,dt\Bigr| &= r = e^{-i\theta}\int_a^b w(t)\,dt = \int_a^b e^{-i\theta}w(t)\,dt \\ &= \mathrm{Re}\Bigl(\int_a^b e^{-i\theta}w(t)\,dt\Bigr) = \int_a^b \mathrm{Re}\bigl(e^{-i\theta}w(t)\bigr)\,dt \\ &\leq \int_a^b \bigl|e^{-i\theta}w(t)\bigr|\,dt \\ &= \int_a^b \bigl|w(t)\bigr|\,dt\end{aligned}$$

□

Standard Properties

Theorem

Suppose that f and g are continuous on a contour Γ.

1. $$\int_\Gamma (f(z)+g(z))\,dz = \int_\Gamma f(z)\,dz + \int_\Gamma g(z)\,dz.$$
2. *If* $\alpha \in \mathbf{C}$, *then* $\alpha \int_\Gamma f(z)\,dz = \int_\Gamma \alpha f(z)\,dz.$
3. *If* $-\Gamma$ *is the opposite contour, then* $$\int_{-\Gamma} f(z)\,dz = -\int_\Gamma f(z)\,dz.$$
4. *If* $|f(z)| \le M$ *for all* $z \in \Gamma$, *then* $$\Big|\int_\Gamma f(z)\,dz\Big| \le M\ell(\Gamma).$$

Proof.

Part (1) is straightforward and (2) is also if we use part (1) of the previous lemma. For part (3), let γ be a directed smooth curve with admissible parameterization $z : [a, b] \to \mathbf{C}$. Then $-\gamma$ is parameterized by $w(s) = z(-s)$ for $s \in [-b, -a]$. Then

$$\begin{aligned}\int_{-\gamma} f(z)\,dz &= \int_{-b}^{-a} f(w(s))w'(s)\,ds \\ &= \int_{-b}^{-a} f(z(-s))(-z'(-s))\,ds\end{aligned}$$

which, after $t = -s$ and $dt = -ds$, is

$$\begin{aligned}&= \int_{b}^{a} f(z(t))z'(t)\,dt = -\int_{a}^{b} f(z(t))z'(t)\,dt \\ &= -\int_{\gamma} f(z)\,dz\end{aligned}$$

Proof.

Now, in general, if $\Gamma = \gamma_1 + \cdots + \gamma_n$, then $-\Gamma = -\gamma_n + \cdots + -\gamma_1$. Then

$$\begin{aligned} \int_{-\Gamma} f(z)\,dz &= \sum_{k=1}^{n} \int_{-\gamma_k} f(z)\,dz \\ &= -\sum_{k=1}^{n} \int_{\gamma_k} f(z)\,dz \\ &= -\int_{\Gamma} f(z)\,dz. \end{aligned}$$

This proves (3).

Proof of (4)

Proof.

We still need to establish (4). But if γ is a directed smooth curve with admissible parameterization $z : [a, b] \to \mathbf{C}$ then

$$\Big|\int_\gamma f(z)\,dz\Big| = \Big|\int_a^b f(z(t))z'(t)\,dt\Big|$$

which, using the lemma, is

$$\begin{aligned} &\le \int_a^b \big|f(z(t))z'(t)\big|\,dt \\ &\le \int_a^b M|z'(t)|\,dt \\ &= M\ell(\gamma). \end{aligned}$$

Now we let $\Gamma = \gamma_1 + \cdots + \gamma_n$, etc. □

An Example

Example

Let C_2 be the positively oriented circle $|z| = 2$. Show that

$$\left| \int_{C_2} \frac{\cos(z)}{z^3+1}\, dz \right| \le \frac{4}{7} e^2 \pi$$

Solution.

If $z = x + iy$ and $|z| = 2$, then
$|\cos(z)| \le \frac{1}{2}\big(|e^{iz}| + |e^{-iz}|\big) = \frac{1}{2}(e^{-y} + e^{y}) \le e^2$. (Since $|y| \le 2$!)
On the the other hand, we can use the reverse triangle inequality to say that $|z^3 + 1| \ge |z|^3 - 1 = 7$ on $|z| = 2$. Thus the integrand above is bounded by $M = \frac{e^2}{7}$. Then the modulus of the integral is bounded by $M\ell(C_2) = \frac{e^2}{7} \cdot 4\pi$. $\square$

Enough

Let's Take a Break