Math 43: Spring 2020
Lecture 11 Part 2

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Find a Better Way

Remark

The first lesson to be learned about evaluating contour integrals is that it is tedious at best and often very difficult as well. But since they are just are just good old line integrals from vector calculus, we know that there should be better ways to evaluate them. That is our first goal today.

A Lemma

Lemma

Let $z : [a, b] \to \mathbf{C}$ be an admissible parameterization of a smooth curve γ in a domain D. Suppose F is analytic in D. Let $w(t) = F(z(t))$ for $t \in [a, b]$. Then w is differentiable and $w'(t) = F'(z(t))z'(t)$.

Proof.

By definition,
$w'(t) = \lim_{h \to 0} \frac{w(t+h)-w(t)}{h} = \lim_{h \to 0} \frac{F(z(t+h))-F(z(t))}{h}$. Since z is one-to-one, $h \neq 0$ implies $z(t+h) - z(t) \neq 0$. Hence
$w'(t) = \lim_{h \to 0} \frac{F(z(t+h))-F(z(t))}{z(t+h)-z(t)} \cdot \frac{z(t+h)-z(t)}{h}$. But as $h \to 0$, $z(t+h) \to z(t)$. Thus $w'(t) =$
$\lim_{w \to z(t)} \frac{F(w)-F(z(t))}{w-z(t)} \cdot \lim_{h \to 0} \frac{z(t+h)-z(t)}{h} = F'(z(t))z'(t)$. $\square$

Fundamental Theorem for Contour Integrals

Theorem (Fundamental Theorem for Contour Integrals)

Suppose that f is continuous on a domain D and that F is an antiderivative for f in D. (That is, $F'(z) = f(z)$ for all $z \in D$.) If Γ is a contour in D from w_1 to w_2, then

$$\int_\Gamma f(z)\,dz = F(w_2) - F(w_1).$$

Proof.

Suppose that γ is a directed smooth curve in D with admissible parameterization $z : [a, b] \to \mathbf{C}$. Then

$$\begin{aligned}\int_\gamma f(z)\,dz = \int_a^b f(z(t))z'(t)\,dt &= \int_a^b \frac{d}{dt}\big(F(z(t))\big)\,dt \\ &= F(z(b)) - F(z(a)).\end{aligned}$$

Proof Continued

Proof Continued.

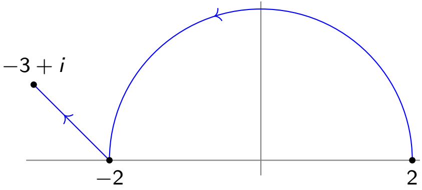
Now suppose that $\Gamma = \gamma_1 + \cdots + \gamma_n$ where γ_k is a directed smooth curve from z_{k-1} to z_k. Then

$$\begin{aligned}
\int_\Gamma f(z)\,dz &= \sum_{k=1}^{n} \int_{\gamma_k} f(z)\,dz \\
&= \sum_{k=1}^{n} F(z_k) - F(z_{k-1}) \\
&= F(z_n) - F(z_0) \\
&= F(w_2) - F(w_1).
\end{aligned}$$

□

Example

Let
$\Gamma = C_2^+ + [-2, -3+i]$. Let's compute $I = \displaystyle\int_\Gamma \cos(z)\,dz$.
Using the definition would be (more than) painful!
But $\frac{d}{dz}\sin(z) = \cos(z)$. Hence
$\boxed{I = \sin(-3+i) - \sin(2)}$.



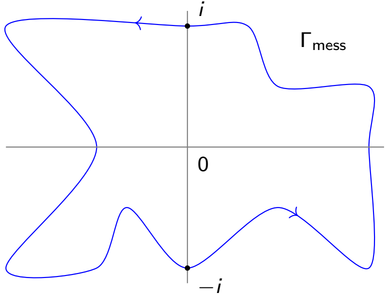
Some Fun

Consider
the complicated contour
Γ_{mess} drawn at right. Then

$$\int_{\Gamma_{\text{mess}}} e^{2z}\,dz = \frac{e^{2z}}{2}\Big|_{i}^{i} = 0.$$

Simple. But what about

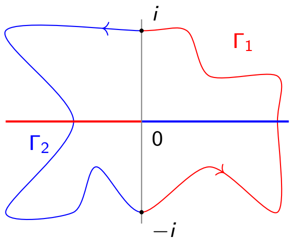
$$\int_{\Gamma_{\text{mess}}} \frac{1}{z}\,dz?$$



We know that there can be no
antiderivative for $\frac{1}{z}$ in a domain that contains the contour Γ_{mess}!
But we can work a bit harder.

We can write $\Gamma_{\text{mess}} = \Gamma_1 + \Gamma_2$ where Γ_1 is the part of Γ_{mess} from $-i$ to i, and Γ_2 is the part from i to $-i$. Then

$$\int_{\Gamma_{\text{mess}}} \frac{1}{z}\,dz = \int_{\Gamma_1} \frac{1}{z}\,dz + \int_{\Gamma_2} \frac{1}{z}\,dz.$$



But Γ_1 sits inside the domain D^* where $\operatorname{Log}(z)$ is analytic. And Γ_2 sits inside the domain D_0^* where $\mathcal{L}_0(z)$ is analtyic. Hence

$$\begin{aligned}\int_{\Gamma_{\text{mess}}} \frac{1}{z}\,dz &= \Big(\operatorname{Log}(z)\Big)\Big|_{-i}^{i} + \Big(\mathcal{L}_0(z)\Big)\Big|_{i}^{-i} \\ &= \operatorname{Log}(i) - \operatorname{Log}(-i) + \mathcal{L}_0(-i) - \mathcal{L}_0(i) \\ &= i\frac{\pi}{2} - (-i\frac{\pi}{2}) + i\frac{3\pi}{2} - i\frac{\pi}{2} = \boxed{2\pi i}.\end{aligned}$$

Enough

That is enough for now!