Math 43: Spring 2020
Lecture 12 and 13 Summary

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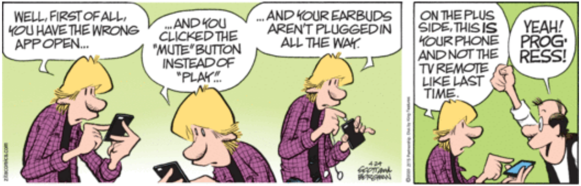
Monday April 27, 2020

Exam Comments

1. I emailed brief exam solutions in a second email. Sunday was a challenging day technology-wise.



2. You can now access a hist-o-gram of the class's scores on the preliminary exam via the assignments page.
3. Take home exams should be—or appear to be—first drafts. I expect polished coherent solutions.
4. You must write in complete sentences.
5. I am happy to discuss the exam in office hours. As a result, you may get put in the "waiting room" while I talk with another student.

Antiderivative Theorem

Theorem (Antiderivative Theorem)

Suppose that $f : D \subset \mathbf{C} \to \mathbf{C}$ is continuous on a domain D. Then the following are equivalent.

1. *f has an antiderivative on D.*
2. *If Γ is any closed contour in D, then*

$$\int_{\Gamma} f(z)\,dz = 0.$$

3. *If Γ_1 and Γ_2 are both contours in D from z_1 to z_2, then*

$$\int_{\Gamma_1} f(z)\,dz = \int_{\Gamma_2} f(z)\,dz.$$

Enough

Remark

While our Antiderivative Theorem gives us a criterion for proving that a continuous function has an antiderivative, it is a big ask. We have to show that every contour integral of f about a closed contour in D is zero. How on earth could we do that!

Deformations

Definition

Suppose that Γ_0 and Γ_1 are closed contours in a domain D. We say that Γ_0 can be continuously deformed to Γ_1 in D if there is a continuous function $z : [0,1] \times [0,1] \to D$ such that

1. For all $s \in (0,1)$, the map $t \mapsto z(s,t)$ is an admissible parameterization of a closed contour Γ_s in D,
2. the map $t \mapsto z(0,t)$ is an admissible parameterization of Γ_0, and
3. the map $t \mapsto z(1,t)$ is an admissible parameterization of Γ_1

Remark

Recall that if Γ_0, Γ_1, or any Γ_s is a point z_0, then we allow the constant function $t \mapsto z_0$ as an admissible parameterization. The idea is that the contours Γ_s "move continuously" staring with Γ_0 and ending at Γ_1.

Idea

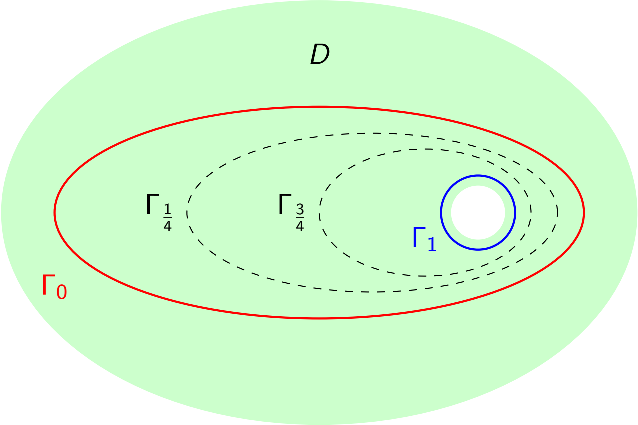


Figure: Continuously deforming Γ_0 into Γ_1 with intermediate contours Γ_s

Simply Connected Domains

Proposition

Let $D = B_r(z_0) = \{ z : |z - z_0| < r \}$ be an open disk. Then every closed contour Γ in D can be continuously deformed to a point in D.

Definition

We say that a domain D is simply connected if every closed contour in D can be continuously deformed to a point in D.

Example

1. The complex plane $D = \mathbf{C}$ is simply connected.
2. Every open disk $D = B_r(z_0)$ is simply connected.
3. The annulus $A = \{ z : 1 < |z| < 2 \}$ is not simply connected.

The Deformation Invariance Theorem

Theorem (Deformation Invariance Theorem)

Suppose that f is analytic in a domain D and that Γ_0 and Γ_1 are closed contours in D such that Γ_0 can be continuously deformed in D to Γ_1. Then

$$\int_{\Gamma_0} f(z)\,dz = \int_{\Gamma_1} f(z)\,dz.$$

In particular, if Γ_0 can be continuously deformed to a point in D, then

$$\int_{\Gamma_0} f(z)\,dz = 0.$$

Cauchy Integral Theorem

Theorem (Cauchy Integral Theorem)

Suppose that f is analytic on a simply connected domain D. Then for every closed contour Γ in D, we have

$$\int_\Gamma f(z)\,dz = 0.$$

Remark

This is the answer to our question, "How on earth do we prove that the contour integral of a function about any closed contour in D is zero"!

Looking Ahead

Example (Using the "Barbell" Contour)

$$\int_{|z|=4} \frac{z^2 - 11z + 22}{(z-2)^2(z+2)}\, dz = 2\pi i.$$

Remark

For Wednesday, we will look at some deeper implications of Cauchy's Integral Theorem and Antiderivatives.