Math 43: Spring 2020
Lecture 12

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Carrying on from Here

- We should be recording this.
- This lecture will be posted today.
- I will also post Monday's lectures today.
- Thus we will be back on a "normal" schedule with the lectures available prior to the actual class time.
- This means that you should be watching the lectures before class time and then starting on the homework for that lecture immediately afterwards.

Antiderivatives

In single-variable calculus, the Fundamental Theorem of Calculus implies that every continuous function has an antiderivative: if $F(x) = \int_a^x f(t)\,dt$ then $F'(x) = f(x)$. However, we know that that situation in our complex world is more complicated. The function $f(z) = 1/z$ does not have an antiderivative in $\mathbf{C} \setminus \{0\}$ while $g(z) = 1/z^n$ most certainly does if $n \geq 2$!

Today we want to investigate conditions under which a continuous function will have an antiderivative in a given domain D. This question turns out to be crucial to some of the really important ideas in the course. If you are really "up" on your vector calculus, you will recognize the similarity to the question of which vector fields have potential functions.

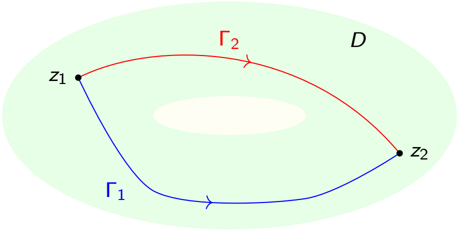
Antiderivative Theorem

Theorem (Antiderivative Theorem)

Suppose that $f : D \subset \mathbf{C} \to \mathbf{C}$ *is continuous on a domain* D*. Then the following are equivalent.*

1. f *has an antiderivative on* D*.*
2. *If* Γ *is any closed contour in* D*, then*
$$\int_\Gamma f(z)\,dz = 0.$$
3. *If* Γ_1 *and* Γ_2 *are both contours in* D *from* z_1 *to* z_2*, then*
$$\int_{\Gamma_1} f(z)\,dz = \int_{\Gamma_2} f(z)\,dz.$$

The Easy Bits



Proof.

(1)$\Longrightarrow$(2): As we observed in Lecture 10, this is an immediate consequence of the Fundamental Theorem for Contour Integrals.

(2)$\Longrightarrow$(3): Let $\Gamma = \Gamma_1 - \Gamma_2$. Then Γ is a closed contour in D. Hence

$$0 = \int_\Gamma f(z)\,dz = \int_{\Gamma_1} f(z)\,dz - \int_{\Gamma_2} f(z)\,dz.$$

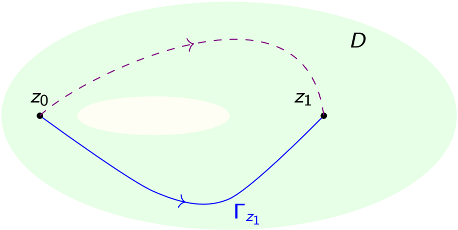
So we need to prove (3)$\Longrightarrow$(1).

The Hard Bit: (3)$\Longrightarrow$(1)

Fix $z_0 \in D$. Let z_1 be any other point in D. Since D is connected there is a contour Γ_{z_1} in D from z_0 to z_1.
Then we can define

$$F(z_1) = \int_{\Gamma_{z_1}} f(z)\,dz.$$

Since we are assuming path independence, the value of $F(z_1)$ does not depend on our choice of Γ_{z_1}. Since z_1 was arbitrary, we can complete the proof by showing that $F'(z_1)$ exists and that $F'(z_1) = f(z_1)$.



Proof Continued

Since D is open, there is a $r > 0$ such that $B_r(z_1) \subset D$. Thus $|w| < r$ implies $z_1 + w \in D$.

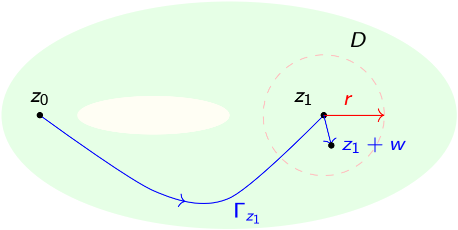
Therefore

$\Gamma_{z_1+w} := \Gamma_{z_1} + [z_1, z_1 + w]$

is a contour from z_0 to $z_1 + w$.

Therefore

$$F(z_1 + w) = \int_{\Gamma_{z_1+w}} f(z)\,dz = \int_{\Gamma_{z_1}} f(z)\,dz + \int_{[z_1, z_1+w]} f(z)\,dz$$



It follows that

$$\frac{F(z_1 + w) - F(z_1)}{w} = \frac{1}{w} \int_{[z_1, z_1+w]} f(z)\,dz.$$

If we parameterize $[z_1, z_1 + w]$ by $\sigma(t) = z_1 + tw$ for $t \in [0, 1]$, then

$$\frac{1}{w} \int_{[z_1, z_1+w]} f(z)\,dz = \frac{1}{w} \int_0^1 f(z_1 + tw) w\,dt = \int_0^1 f(z_1 + tw)\,dt$$

The Rest

We have
$$\frac{F(z_1+w)-F(z_1)}{w} - f(z_1) = \frac{1}{w}\int_{[z_1,z_1+w]} f(z)\,dz - f(z_1) =$$
$$\int_0^1 f(z_1+tw)\,dt - f(z_1) = \int_0^1 \big[f(z_1+tw)-f(z_1)\big]\,dt.$$ If $\epsilon > 0$, then the continuity of f implies that we can take $r > 0$ small enough so that $|w| < r$ implies that $\big|f(z_1+tw)-f(z_1)\big| < \epsilon$ for all $t \in [0,1]$. Then

$$\Big|\frac{F(z_1+w)-F(z_1)}{w} - f(z_1)\Big| \leq \int_0^1 \big|f(z_1+tw)-f(z_1)\big|\,dt \leq \epsilon.$$

It follows that $\lim_{w\to 0} \frac{F(z_1+w)-F(z_1)}{w} - f(z_1) = 0$ and hence that $F'(z_1) = f(z_1)$.

That completes the proof. $\square$

Enough

Remark

While our Antiderivative Theorem gives us a criterion for proving that a continuous function has an antiderivative, it is a big ask. We have to show that every contour integral of f about a closed contour in D is zero. How on earth could we do that! Surprisingly, we will have a good answer, but that will have to wait until next week.

You've had a busy week! So we can all have a rest, I think that is enough until next week!