Math 43: Spring 2020
Lecture 13 Part I

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What Now

Remark

Our next goal is a suite of results referred to as *Cauchy's Integral Theorem* and *Cauchy's Integral Formula*. These will be the first truly significant results of the course and will lead to some extraordinary results. Our text allows two separate paths to establish these results, and we will be following approach using deformations of contours as in section 4.4a. The other approach using vector calculus and Green's Theorem in particular is covered in section 4.4b. You will not be responsible for section 4.4b.

Deformations

Definition

Suppose that Γ_0 and Γ_1 are closed contours in a domain D. We say that Γ_0 can be continuously deformed to Γ_1 in D if there is a continuous function $z : [0,1] \times [0,1] \to D$ such that

1. For all $s \in (0,1)$, the map $t \mapsto z(s,t)$ is an admissible parameterization of a closed contour Γ_s in D,
2. the map $t \mapsto z(0,t)$ is an admissible parameterization of Γ_0, and
3. the map $t \mapsto z(1,t)$ is an admissible parameterization of Γ_1

Remark

Recall that if Γ_0, Γ_1, or any Γ_s is a point z_0, then we allow the constant function $t \mapsto z_0$ as an admissible parameterization. The idea is that the contours Γ_s "move continuously" staring with Γ_0 and ending at Γ_1.

Idea

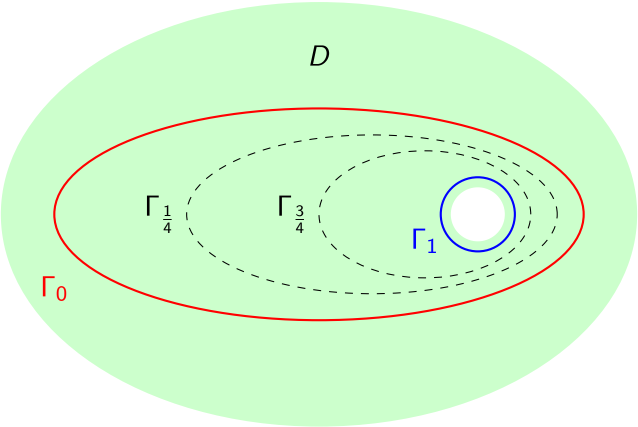
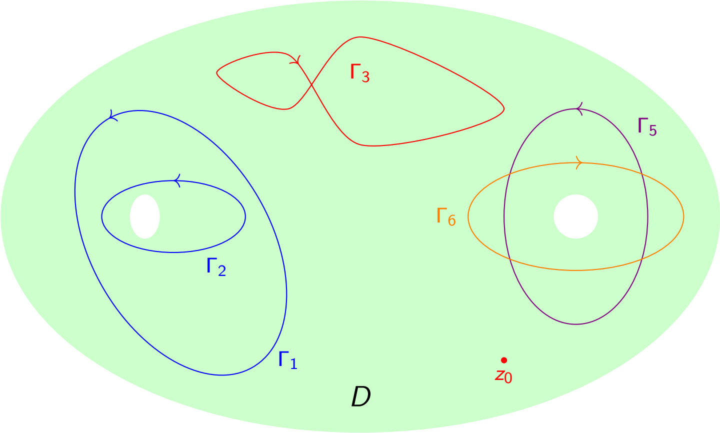


Figure: Continuously deforming Γ_0 into Γ_1 with intermediate contours Γ_s

Examples



Because of the holes in D, our intuition tells us that we can deform Γ_1 and Γ_2 to one another. We can deform Γ_3 to the contour consisting of the point z_0 and vice versa. However we cannot deform Γ_5 and Γ_6 to one another. Nor are any other deformations between these contours possible.

The Barbell Contour

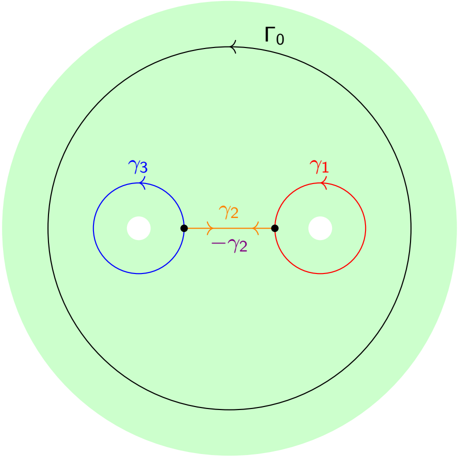


Figure: Consider deforming the contour Γ_0 given by the positively oriented circle $|z| = 4$ in side D to the "bar-bell" contour $\Gamma_1 = \gamma_1 + \gamma_2 + \gamma_3 - \gamma_2$ with the given orientations. Notice we're allowing D to omit points inside the circles γ_1 and γ_3. (See the Mathematica Demo!)

A Formal Example

Proposition

Let $D = B_r(z_0) = \{ z : |z - z_0| < r \}$ be an open disk. Then every closed contour Γ in D can be continuously deformed to a point in D.

Proof.

Let $w : [0,1] \to D$ be an admissible parameterization of Γ. Define $z : [0,1] \times [0,1] \to \mathbf{C}$ by

$$z(s,t) = sz_0 + (1-s)w(t).$$

The only issue to check is that $z(s,t) \in D$ for all (s,t)! But

$$\begin{aligned} |z(s,t) - z_0| &= |sz_0 + (1-s)w(t) - z_0| = |(1-s)(w(t) - z_0)| \\ &\leq |w(t) - z_0| < r. \end{aligned}$$

□

Simply Connected Domains

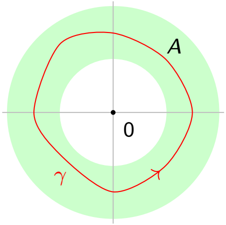
Definition

We say that a domain D is simply connected if every closed contour in D can be continuously deformed to a point in D.

Example

1. The complex plane $D = \mathbf{C}$ is simply connected.
2. Every open disk $D = B_r(z_0)$ is simply connected.
3. The annulus $A = \{ z : 1 < |z| < 2 \}$ is not simply connected.

While it is intuitively clear that A is not simply connected, like the Jordan Curve Theorem, it is not so easy to give a careful proof as we did for open disks. Nevertheless, will be able to prove this soon.



Enough

Remark

So what good are deformations except for drawing pretty pictures. We will see in the second part of the lecture.

Time for a Break