Math 43: Spring 2020
Lecture 13 Part II

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The Deformation Invariance Theorem

Theorem (Deformation Invariance Theorem)

Suppose that f is analytic in a domain D and that Γ_0 and Γ_1 are closed contours in D such that Γ_0 can be continuously deformed in D to Γ_1. Then

$$\int_{\Gamma_0} f(z)\,dz = \int_{\Gamma_1} f(z)\,dz.$$

In particular, if Γ_0 can be continuously deformed to a point in D, then

$$\int_{\Gamma_0} f(z)\,dz = 0.$$

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Remark

The Deformation Invariance Theorem is one of the deeper results we'll discuss this term. Unfortunately, the proof for the result as stated is a bit more than we want to take on in Math 43. Instead, we will prove it with some additional assumptions. Namely,

1. f' is continuous on D.
2. The deformation $z : [0, 1]^2 \to D$ from Γ_0 to Γ_1 has continuous second partials throughout D.
3. Some more Calculus.

The Proof of DIT

Proof.

Let Γ_s be the contour $t \mapsto z(s,t)$ for $t \in [0,1]$. Define $I(s) = \int_{\Gamma_s} f(z)\,dz = \int_0^1 f(z(s,t))z_t(s,t)\,dt$. We need to prove that $I(0) = I(1)$. Hence it suffices to show that for all $s \in [0,1]$,

$$\begin{aligned} I'(s) &= 0 \\ &= \frac{d}{ds}\int_0^1 f(z(s,t))z_t(s,t)\,dt \end{aligned}$$

which, assuming we can differentiate under the integral sign, is

$$= \int_0^1 \frac{\partial}{\partial s}\big[f(z(s,t))z_t(s,t)\big]\,dt.$$

Proof

Proof Continued.

However using the chain rule:

$$\int_0^1 \frac{\partial}{\partial s}\big[f(z(s,t))z_t(s,t)\big]\,dt$$
$$= \int_0^1 \Big[f'(z(s,t))z_s(s,t)z_t(s,t) + f(z(s,t))z_{ts}(s,t)\Big]$$

which, since $z_{ts} = z_{st}$ by Clairaut's Theorem, is

$$= \int_0^1 \frac{\partial}{\partial t}\big(f(z(s,t))z_s(s,t)\big)\,dt$$

which, by the Fundamental Theorem of Calculus, is

$$= f(z(s,1))z_s(s,1) - f(z(s,0))z_s(s,0).$$

Proof Continued.

But each Γ_s is a closed contour, so $z(s,1) = z(s,0)$. Similarly, $s \mapsto z(s,0)$ and $s \mapsto z(s,1)$ are the same function. Therefore $z_s(s,0) = z_s(s,1)$. Therefore,

$$\begin{aligned} I'(s) &= f(z(s,1))z_s(s,1) - f(z(s,0))z_s(s,0) \\ &= 0. \end{aligned}$$

Therefore $s \mapsto I(s)$ is constant and $I(0) = I(1)$ and we're done. $\square$

Cauchy Integral Theorem

Theorem (Cauchy Integral Theorem)

Suppose that f is analytic on a simply connected domain D. Then for every closed contour Γ in D we have

$$\int_{\Gamma} f(z)\,dz = 0.$$

Proof.

Suppose Γ is a closed contour in D. Since D is simply connected, we can continuously deform Γ to a point. Hence the result follows from the Deformation Invariance Theorem. $\square$

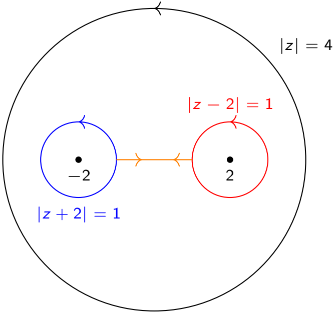
Back to the Barbell

Example

Evaluate $I = \displaystyle\int_{|z|=4} \frac{z^2 - 11z + 22}{(z-2)^2(z+2)}\,dz$. Where we assume $|z| = 4$ is positively oriented.

Note that $f(z) := \dfrac{z^2 - 11z + 22}{(z-2)^2(z+2)} = \dfrac{1}{(z-2)^2} - \dfrac{2}{z-2} + \dfrac{3}{z+2}$.

We can deform $|z| = 4$ to a barbell contour as before. Since the line segments cancel, $I = \displaystyle\int_{|z-2|=1} f(z)\,dz + \int_{|z+2|=1} f(z)\,dz$.



Example Continued

We will take the integrals one at a time.

$$\begin{aligned}
\int_{|z-2|=1} f(z)\,dz &= \int_{|z-2|=1} \Big(\frac{1}{(z-2)^2} - \frac{2}{z-2} + \frac{3}{z+2} \Big)\,dz \\
&= \int_{|z-2|=1} (z-2)^{-2}\,dz - 2\int_{|z-2|=1} (z-2)^{-1}\,dz \\
&\qquad\qquad + 3\int_{|z-2|=1} (z+2)^{-1}\,dz \\
&= \underbrace{0 - 2(2\pi i)}_{\text{Basic Circle Lemma}} + \underbrace{0}_{\text{Cauchy's Integral Theorem}} \\
&= -4\pi i.
\end{aligned}$$

The Other Bit

Similarly,

$$\int_{|z+2|=1} f(z)\,dz = \int_{|z+2|=1} \Big(\frac{1}{(z-2)^2} - \frac{2}{z-2} + \frac{3}{z+2} \Big)\,dz$$
$$= \underbrace{0+0}_{\text{Cauchy's Integral Theorem}} + 6\pi i.$$

Hence $I = -4\pi i + 6\pi i = \boxed{2\pi i}$.

Remark

Next time we will look at some deeper implications of Cauchy's Integral Theorem and Antiderivatives.

That's Enough for Today