Math 43: Spring 2020
Lecture 14 Part I

Dana P. Williams

Dartmouth College

Wednesday April 29, 2020

Last Time

As a result of the Deformation Invariance Theorem, we were able to prove the following.

Theorem (Cauchy Integral Theorem)

Suppose that f is analytic on a simply connected domain D and that Γ is a closed contour in D. Then

$$\int_\Gamma f(z)\,dz = 0.$$

Remark

Note the requirements:

1. D is simply connected.
2. f is analytic on D.
3. Γ is closed and contained in D.

Theorem

Suppose that D is a simply connected domain. Then every analytic function f on D has an antiderivative on D.

Proof.

Suppose f is analytic on D. Then the Cauchy Integral Theorem implies $\int_\Gamma f(z)\,dz = 0$ for every closed contour in D. Hence f has an antiderivative by our Antiderivative Theorem. □

A Promise Kept

Corollary

Neither the punctured plane $D' = \mathbf{C} \setminus \{0\}$ *nor the annulus* $A = \{\, z : 1 < |z| < 2 \,\}$ *is simply connected.*

Proof.

$f(z) = 1/z$ is analytic in both domains, but
$$\int_{|z|=\frac{3}{2}} \frac{1}{z}\,dz = 2\pi i \neq 0.$$
$\square$

Simply Connected Domains

Remark

Recall that the Jordan Curve Theorem says that every simple closed contour Γ in the plane divides the plane into two domains. Furthermore, only one of these is bounded and it is called the interior of Γ.

Theorem (The Jordan Curve Theorem Revisited)

Suppose that Γ is a simple closed contour. (We may call such a Γ a Jordan contour.) Then the interior of Γ is simply connected.

Remark

The proof is beyond the scope of this course, but we will use it to provide additional examples of simply connected regions.

Example

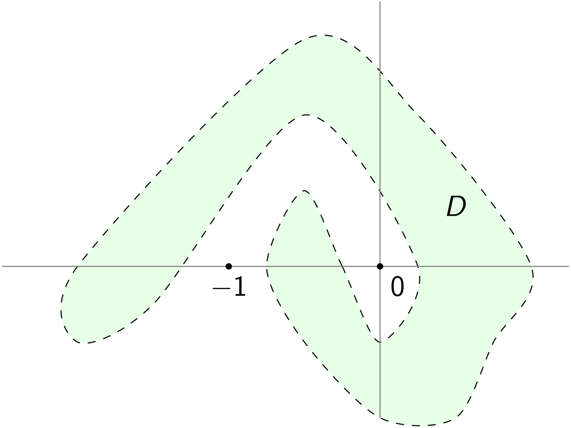


Figure: A Simply Connected Domain D

Branches

Corollary

Suppose that D is a simply connected domain that does not contain 0. Then there is an analytic branch of $\log(z)$ in D.

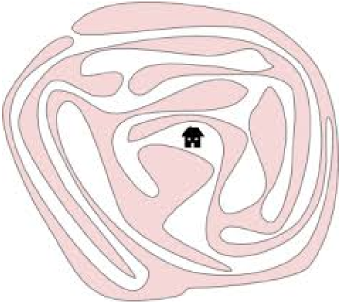
Proof.

Since $0 \notin D$, $f(z) = \frac{1}{z}$ is analytic on D. Since D is simply connected, f has an antiderivative g on D. That is, $g'(z) = \frac{1}{z}$ for all $z \in D$. Now let $h(z) = \dfrac{e^{g(z)}}{z}$. Since $0 \notin D$, h is analytic on D and $h'(z) = \dfrac{g'(z)e^{g(z)}z - e^{g(z)}}{z^2} = 0$ for all $z \in D$. Therefore h is constant. Since h is nonzero, there is a nonzero complex number c such that $ce^{g(z)} = z$ for all $z \in D$. Let $d \in \log(c)$. Then $e^d = c$. Now let $h(z) = g(z) + d$. Then $e^{h(z)} = e^{d+g(z)} = e^d e^{g(z)} = ce^{g(z)} = z$. Thus h is analytic on D and $h(z) \in \log(z)$ for all $z \in D$. □

Exotic Branches

Example

This means that there is an analytic branch of $\log(z)$ in the domain D from two slides previous! Even better, if the house in the picture below sits on the origin, then there is an analytic branch of $\log(z)$ defined in the pink domain below.



Another Corollary

Corollary

Suppose that u is harmonic in a simply connected domain D. Then u has a harmonic conjugate in D.

Proof.

We are asked to find v such that $f(z) = u(z) + iv(z)$ is analytic on D. If such a v exists, then $f'(z) = u_x(z) + iv_x(z) = u_x(z) - iu_y(z)$. Let $g(z) = u_x(z) - iu_y(z)$. Note that

$$g_x(z) = u_{xx}(z) - iu_{yx}(z) = -i\big(u_{yx}(z) - iu_{xx}(z)\big)$$

which, by Clairaut's Theorem and since u is harmonic, is

$$= -i\big(u_{xy}(z) + iu_{yy}(z)\big) = -ig_y(z).$$

Since u has continuous second partials, g has continuous first partials which satisfy the Cauchy-Riemann Equations. Therefore, g is analytic in D.

Proof.

Since $g(z) = u_x(z) - iu_y(z)$ is analytic in a simply connected domain D, g has a antiderivative h in D. Let $h(z) = \underline{u}(z) + i\underline{v}(z)$. Then $g(z) = h'(z) = \underline{u}_x(z) + i\underline{v}_x(z) = \underline{u}_x(z) - i\underline{u}_y(z)$. Let $w := u - \underline{u}$. Then $w_x = u_x - \underline{u}_x \equiv 0$ while $w_y = u_y - \underline{u}_y \equiv 0$. Therefore w is a constant function. Therefore $\underline{u}(z) = u(z) + c$. Then if we let $f(z) = h(z) - c$, then $f(z) = u(z) + i\underline{v}(z)$ is analytic in D, and $\underline{v}$ is a harmonic conjugate for u. $\square$

Enough

Time for a Break