Math 43: Spring 2020
Lecture 14 Part II

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What Next

Remark

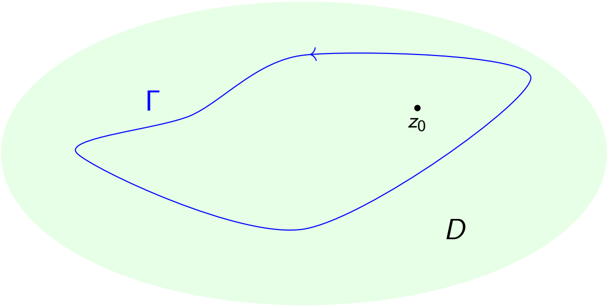
The Cauchy Integral Theorem is a fundamental result. It underlies much of what we're going to do. For example, it allows to assert that analytic functions on simply connected domains have antiderivatives. Now we are going to take on a companion result whose computational value actually hides its theoretical significance.

The Cauchy Integral Formula

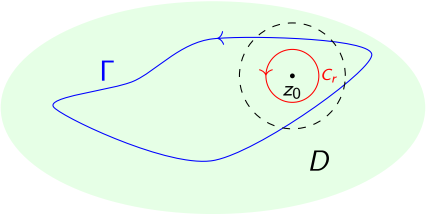
Theorem (Cauchy Integral Formula)

Suppose that Γ is a positively oriented simple closed contour in a simply connected domain D. If z_0 lies inside of Γ and if f is analytic in D, then

$$f(z_0) = \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w - z_0}\, dw.$$



Proof



Let $g(w) = \frac{f(w)}{w-z_0}$. Then g is analytic in $D' = D \setminus \{z_0\}$. Since D is open, there is a $r > 0$ such that $B_{2r}(z_0) \subset D$. In particular, the circle C_r of radius r centered at z_0 is contained in D.

We take it as "clear" that we can continuously deform Γ into C_r inside the deleted domain D'. Hence by the DIT

$$\int_\Gamma \frac{f(w)}{w-z_0}\,dw = \int_{C_r} \frac{f(w)}{w-z_0}\,dz = \underbrace{f(z_0)\int_{C_r} \frac{1}{w-z_0}\,dw}_{A} + \underbrace{\int_{C_r} \frac{f(w)-f(z_0)}{w-z_0}\,dw}_{B}.$$

By the Basic Circle Lemma, $A = 2\pi i \cdot f(z_0)$.

Thus to complete the proof, we need to see that $B = 0$!

Computing B

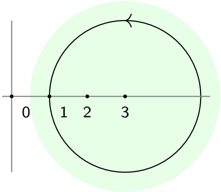
Notice that if $0 < \delta \le r$, then the DIT implies that

$$B = \int_{C_r} \frac{f(w) - f(z_0)}{w - z_0}\, dw = \int_{C_\delta} \frac{f(w) - f(z_0)}{w - z_0}$$

where C_δ is the circle of radius δ centered at z_0. If $M_\delta := \max\{\, |f(w) - f(z_0)| : w \in C_\delta \,\}$, then we have

$$|B| \le \frac{M_\delta}{\delta} \cdot 2\pi\delta = 2\pi M_\delta.$$

Since f is continuous at z_0, we must have $\lim_{\delta \searrow 0} M_\delta = 0$. Since the above estimate holds for any $0 < \delta \le r$, we must have $|B| = 0$.



Example

Let Γ be the positively oriented circle $|z-3|=2$. Evaluate $I=\displaystyle\int_\Gamma \frac{\cos(z^2)}{z^2(z-2)}\,dz$

Solution.

Notice that $f(z)=\dfrac{\cos(z^2)}{z^2}$ is analytic in the simply connected domain $B_{\frac{5}{2}}(3)=\{\, z : |z-3|<\frac{5}{2}\,\}$. Hence we can apply the Cauchy Integral Formula to the integral $\displaystyle\int_\Gamma \frac{f(w)}{w-2}\,dw$. Therefore

$$I=2\pi i f(2)=2\pi i\cdot\frac{\cos(4)}{4}=\boxed{\frac{\pi i\cos(4)}{2}}.$$

□

Some Shortcuts

Remark (Just Between Us)

The authors like the phase, "suppose that f is analytic on and inside a simply closed contour Γ". This means that f is analytic on the domain that forms the interior of Γ and at each point of Γ. Recall the technicality that we say f is analytic at a point only if it is analytic in a neighborhood of that point. It follows that if "f is analytic on and inside a simply closed contour Γ", then f is analytic in domain that contains Γ. We are going to accept the highly nontrivial fact that we can take this domain to be simply connected. This is what we did explicitly in the previous example where $f(z) = \cos(z^2)/z^2$ was clearly analytic on and inside $|z-3| = 2$.

NC-17 Versions

The assumption we've made on the previous slide allow us to prove the following NC-17 versions of the Cauchy Integral Theorem and the Cauchy Integral Formula.

Theorem (Cauchy NC-17)

Suppose that f is analytic on and inside a simple closed contour Γ.

1. *Then*
$$\int_\Gamma f(z)\,dz = 0,$$
2. *and if z_0 lies inside of Γ, then*
$$f(z_0) = \frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{w - z_0}\,dw.$$

Proof.

As per the previous remark, we can assume that there is a simply connected domain D containing Γ. Then these assertions follow from the Cauchy Theorems as previously stated. □

Time for a Break