Math 43: Spring 2020
Lecture 15 Summary

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Notes

1. Turn on the recording.
2. There is a pre-recorded lecture for Monday, May 4th, as well as a homework assignment.
3. However there will be no regular class meeting Monday morning.
4. Instead, I will have office hours (same zoom coordinates) from 1:30-2:30.
5. Of course, I will also have my usual TTh office hours.
6. On Wednesday, we will summarize both Monday and Wednesday's lectures.
7. I'd like us—that is you—to start a canvas discussion on the best techniques for watching and learning from the recorded lectures.
8. I've posted a response from "Test Student" titled "How I watch the videos" that you reply to. Our you can post your own. We all might learn something.

Riemann's Theorem

Theorem (Riemann's Theorem)

Suppose that g *is continuous on a contour* Γ. *Let* $D = \{ z : z \notin \Gamma \}$. *For all* $n \in \mathbf{N}$, *let*

$$F_n(z) = \int_\Gamma \frac{g(w)}{(w-z)^n}\, dw.$$

Then F_n *is analytic in* D *and*

$$F_n'(z) = nF_{n+1}(z) = n\int_\Gamma \frac{g(w)}{(w-z)^{n+1}}\, dw.$$

Some Comments

1. If we believed we could pass the complex derivative under the integral sign, we'd immediately have $F_n'(z) = nF_{n+1}(z)$. This isn't a proof, but it at least makes it easy to remember the formula.
2. g only need be continuous on Γ.
3. Γ does not have to be closed.
4. D is open, but is usually not a domain.
5. The proof was a bit of work.

The Big Result

Theorem

Suppose that f is analytic on a domain D. Then f' is also analytic on D.

Corollary

Suppose that f is analytic on a domain D. Then f is smooth on D. That is, $f^{(n)}(z)$ exists for all $n \in \mathbf{N}$ and $z \in D$.

Cauchy's Formula for the Derivatives

Theorem (Cauchy's Formula for the Derivatives)

Suppose that f is analytic on and inside a simple closed contour Γ. Then for all z lying inside of Γ,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(w)}{(w-z)^{n+1}}\, dz \quad \textit{for } n = 0, 1, 2, dots.$$

More Miracles

Corollary

Suppose that $f(z) = u(z) + iv(z)$ is analytic in a domain D. Then u and v have continuous partials of all orders in D. In particular, u and v are always harmonic.

Corollary

Suppose that u is harmonic on a domain D. Then u has continuous partials of all orders.

A Converse to Cauchy's Integral Theorem

Theorem (Moreara's Theorem)

Suppose that f is continuous on a domain D. If

$$\int_\Gamma f(z)\,dz = 0$$

for every closed contour Γ in D, then f is analytic in D.

Remember

Remember that there will be NO class meeting on Monday, May 4th. However, there will be a recorded lecture for Monday as usual. I will have office hours Monday from 1:30-2:30 if you have questions, and we will review Monday's as well as Wednesday's lecture on Wednesday during our class meeting time.