Math 43: Spring 2020
Lecture 15 Part I

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More That Meets the Eye

Remark

While the Cauchy Integral Formula seems more computational than theoretical, it has some remarkable consequences that aren't immediately apparent. To mine those, we need what I call Riemann's Theorem. The rump version of this in the text is not sufficient to derive the really cool results. So we will pay the piper, and give a proper proof. Hold onto your socks.

Theorem (Riemann's Theorem)

Suppose that g is continuous on a contour Γ. Let $D = \{ z : z \notin \Gamma \}$. For all $n \in \mathbf{N}$, let

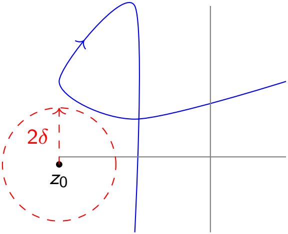
$$F_n(z) = \int_\Gamma \frac{g(w)}{(w - z)^n} \, dw.$$

Then F_n is analytic in D and

$$F_n'(z) = nF_{n+1}(z) = n \int_\Gamma \frac{g(w)}{(w - z)^{n+1}} \, dw.$$

Some Comments

1. If we believed we could pass the complex derivative under the integral sign, we'd immediately have $F_n'(z) = nF_{n+1}(z)$. This isn't a proof, but it at least makes it easy to remember the formula.
2. g only need be continuous on Γ.
3. Γ does not have to be closed.
4. D is open, but is usually not a domain.
5. The proof is challenging.



Fix $z_0 \in D$. Since D is open, there is a $\delta > 0$ such that $B_{2\delta}(z_0) \subset D$. Thus $|w - z_0| \geq 2\delta$ if $w \in \Gamma$. Note that

$$\frac{1}{w-z} - \frac{1}{w-z_0} = \frac{z-z_0}{(w-z)(w-z_0)}.$$

Therefore, using the red equation,

$$\begin{aligned} \big|F_1(z) - F_1(z_0)\big| &= \Big|\int_\Gamma \Big(\frac{1}{w-z} - \frac{1}{w-z_0}\Big) g(w)\,dw\Big| \\ &= |z - z_0| \Big|\int_\Gamma \frac{g(w)}{(w-z)(w-z_0)}\,dw\Big|. \end{aligned}$$

Continued

Let $M = \max\{ |g(w)| : w \in \Gamma \}$. Suppose that $0 < |z - z_0| < \delta$. Then if $w \in \Gamma$,

$$|w - z| \geq |w - z_0| - |z - z_0| \geq \delta.$$

Therefore

$$\big|F_1(z) - F_1(z_0)\big| \leq \frac{M}{\delta \cdot 2\delta} \ell(\Gamma)|z - z_0|.$$

It follows that

$$\lim_{z \to z_0} \big|F_1(z) - F_1(z_0)\big| = 0.$$

Therefore F_1 is continuous at z_0. Since $z_0 \in D$ was arbitrary, F_1 is continuous on D.

And Now $G_n(z)$

Now we get clever and introduce a new function

$$\begin{aligned} G_n(z) &:= \int_\Gamma \frac{g(w)}{(w-z)^n(w-z_0)}\,dw \\ &= \int_\Gamma \frac{\tilde{g}(w)}{(w-z)^n}\,dw, \end{aligned}$$

where $\tilde{g}(w) = g(z)/(z-z_0)$.

Since $\tilde{g}$ is also continuous on Γ, it follows that G_1 is continuous on D.

Notice also that

$$G_n(z_0) = F_{n+1}(z_0)!$$

Why Introduce G_n?

Note that

$$\frac{F_1(z) - F_1(z_0)}{z - z_0} = \frac{1}{z - z_0} \int_\Gamma \Big(\frac{1}{w - z} - \frac{1}{w - z_0} \Big) g(w)\, dw$$
$$= \int_\Gamma \frac{g(w)}{(w - z)(w - z_0)}\, dw = G_1(z).$$

Now we notice that, since G_1 is continuous,

$$F_1'(z_0) = \lim_{z \to z_0} \frac{F_1(z) - F_1(z_0)}{z - z_0} = \lim_{z \to z_0} G_1(z) = G_1(z_0) = F_2(z_0).$$

This proves the result when $n = 1$.

The Inductive Hypothesis

We now proceed by induction. We assume we have proved the result

$$F'_{n-1}(z) = (n-1)F_n(z) \quad \text{for } n \geq 2.$$

We have to prove that $F'_n(z) = nF_{n+1}(z)$. But we can replace $g(w)$ with $\tilde{g}(w) = g(z)/(z - z_0)$ in the above so that we also have

$$G'_{n-1}(z) = (n-1)G_n(z).$$

Keep on Truckin'

Since $\dfrac{1}{w-z} = \dfrac{1}{w-z_0} + \dfrac{z-z_0}{(w-z)(w-z_0)}$,

$$\begin{aligned} F_n(z) &= \int_\Gamma \frac{g(w)}{(w-z)^n}\,dw = \int_\Gamma \frac{g(w)}{(w-z)^{n-1}(w-z)}\,dw \\ &= \int_\Gamma \Big[\frac{g(w)}{(w-z)^{n-1}(w-z_0)} + \frac{g(w)(z-z_0)}{(w-z)^n(w-z_0)}\Big]\,dw \\ &= G_{n-1}(z) + (z-z_0)G_n(z). \end{aligned}$$

Therefore

$$\begin{aligned} F_n(z) - F_n(z_0) &= F_n(z) - G_{n-1}(z_0) \\ &= G_{n-1}(z) - G_{n-1}(z_0) + (z-z_0)G_n(z). \end{aligned}$$

Getting There

Note that if $0 < |z - z_0| < \delta$, then

$$|G_n(z)| = \left| \int_\Gamma \frac{g(w)}{(w-z)^n(w-z_0)}\,dw \right| \leq \frac{M}{\delta^n \cdot 2\delta}\ell(\Gamma).$$

Also, since G_{n-1} is differentiable, it is continuous. Therefore, using the last slide,

$$\begin{aligned} \bigl|F_n(z) - F_n(z_0)\bigr| &\leq \bigl|G_{n-1}(z) - G_{n-1}(z_0)\bigr| + |z - z_0||G_n(z)| \\ &\leq \bigl|G_{n-1}(z) - G_{n-1}(z_0)\bigr| + |z - z_0|\frac{M\ell(\Gamma)}{\delta^n \cdot 2\delta} \end{aligned}$$

Therefore $\lim_{z \to z_0} \bigl|F_n(z) - F_n(z_0)\bigr| = 0$, and F_n is continuous. Hence G_n is continuous as well.

QED

Now we have

$$\begin{aligned} F_n'(z) &= \lim_{z\to z_0} \frac{F_n(z)-F_n(z_0)}{z-z_0} \\ &= \lim_{z\to z_0}\Bigl(\frac{G_{n-1}(z)-G_{n-1}(z_0)}{z-z_0}+G_n(z)\Bigr) \\ &= G_{n-1}'(z_0)+G_n(z_0) \\ &= (n-1)G_n(z_0)+G_n(z_0) \\ &= nG_n(z_0) \\ &= nF_{n+1}(z_0) \end{aligned}$$

And we're done! $\square$

Enough

Now *that* is an induction proof! Let's take a break. After we rest up, we'll see why Riemann's Theorem was worth all the effort.