Math 43: Spring 2020
Lecture 15 Part II

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Friday May 1, 2020

A Little Riemann in your Day

Remark

It is time to see why we worked so hard to prove Riemann's Theorem. Recall that in calculus of real variables, not all differentiable functions are smooth. In particular, the derivative of a function need not even be continuous let alone differentiable.

The function

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & \text{if } x \neq 0\text{, and} \\ 0 & \text{if } x = 0 \end{cases}$$

is differentiable with derivative

$$f'(x) = \begin{cases} 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & \text{if } x \neq 0\text{, and} \\ 0 & \text{if } x = 0. \end{cases}$$

The Big Result

Theorem

Suppose that f is analytic on a domain D. Then f' is also analytic on D.

Proof.

We just need to see that $f''(z_0)$ exists for all $z_0 \in D$. Since D is open, there is a $r > 0$ such that $B_{2r}(z_0) \subset D$. Hence, if C_r is the positively oriented circle C_r of radius r centered at z_0, then f is analytic on and inside C_r. Hence the Cauchy Integral Formula implies that for all $z \in B_r(z_0)$ (which is the interior of C_r) $f(z) = \frac{1}{2\pi i} F_1(z)$ where $F_n(z) = \int_{C_r} \frac{f(w)}{(w-z)^n}\,dz$. By Riemann's Theorem, F_2 is analytic on $B_r(z_0)$. Moreover $f'(z) = \frac{1}{2\pi i} F_1'(z) = \frac{1}{2\pi i} F_2(z)$. Therefore f' is analytic on $B_r(z_0)$. In particular, $f''(z_0) = \frac{1}{2\pi i} F_2'(z_0) = \frac{1}{\pi i} F_3(z_0)$. $\square$

Analytic Functions are smooth

We just proved that if f is analytic on D, then so is its derivative f'. Of course, then f'' exists and is itself analytic. Therefore an analytic function f must have derivatives of all orders. It is standard to write $f^{(n)}$ for the n^{th} derivative of f and to call a function smooth if it has derivatives of all orders. We have proved the following corollary.

Corollary

Suppose that f is analytic on a domain D. Then f is smooth on D. That is, $f^{(n)}(z)$ exists for all $n \in \mathbf{N}$ and $z \in D$.

Cauchy's Formula for the Derivatives

Theorem (Cauchy's Formula for the Derivatives)

Suppose that f is analytic on and inside a simple closed contour Γ. Then for all z lying inside of Γ,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(w)}{(w-z)^{n+1}}\, dz \quad \text{for } n = 0, 1, 2, dots.$$

Proof.

Let $F_n(z) = \displaystyle\int_\Gamma \frac{f(w)}{(w-z)^n}\, dw$. By the CIF, $F_1(z) = 2\pi i \cdot f(z)$. By Riemann's Theorem, F_1 has derivatives of all orders. In fact, I claim $F_1^{(n)}(z) = n! F_{n+1}(z)$. This is clearly true if $n = 1$. If we assume that $F_1^{(n-1)}(z) = (n-1)! F_n(z)$, then
$F_1^{(n)}(z) = (n-1)! F_n'(z) = (n-1)! n F_{n+1}(z) = n! F_{n+1}(z)$ as claimed.
But then

$$2\pi i \cdot f^{(n)}(z) = F_1^{n}(z) = n! F_{n+1}(z) = n! \int_\Gamma \frac{f(w)}{(w-z)^{n+1}}\, dw.$$

This completes the proof. □

More Miracles

Corollary

Suppose that $f(z) = u(z) + iv(z)$ is analytic in a domain D. Then u and v have continuous partials of all orders in D. In particular, u and v are always harmonic.

Proof.

Since $v = \operatorname{Re}(-if)$, it suffices to consider u. Recall that we can use our Cauchy-Riemann Theorems to assert that $f'(z) = u_x(z) - iu_y(z)$. Since f' is analytic, it is continuous. Hence the first partials of u are continuous. But f' is analytic so on the one hand

$$f''(z) = \frac{\partial}{\partial x}\big(u_x(z) - iu_y(z)\big) = u_{xx}(z) - iu_{yx}(z).$$

On the other hand

$$f''(z) = -i\frac{\partial}{\partial y}\big(u_x(z) - iu_y(z)\big) = -u_{yy}(z) - iu_{xy}(z).$$

Hence all of u's second partials exist and are continuous. Since $u_x = \operatorname{Re}(f')$ and $u_y = \operatorname{Re}(if')$, we see that all u's third partials exist and are continuous. Etc. $\square$

Even Better

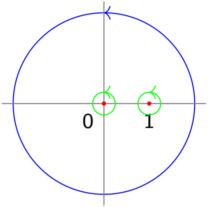
Corollary

Suppose that u is harmonic on a domain D. Then u has continuous partials of all orders.

Proof.

Let $z_0 \in D$. It suffices to see that u has continuous partials of all orders at z_0. (Being harmonic is a local property!) Since D is open, there is a $r > 0$ such that $B_r(z_0) \subset D$. Since $B_r(z_0)$ is simply connected, u has a harmonic conjugate v in $B_r(z_0)$. That is, $f(z) = u(z) + iv(z)$ is analytic on $B_r(z_0)$. But the previous result then implies that u has continuous partials of all orders inside $B_r(z_0)$. This suffices. □

An Example



Example

Evaluate $I = \int_{|z|=2} \frac{e^{3z}}{z(z-1)^3}\,dz$ (where we have oriented $|z| = 2$ positively).

We can use the DIT to deform $|z| = 2$ into two positively oriented circles $|z| = \frac{1}{4}$ and $|z-1| = \frac{1}{4}$ (together with two line segments whose contributions cancel). Hence

$$I = \int_{|z|=\frac{1}{4}} \frac{g(z)}{z}\,dz + \int_{|z-1|=\frac{1}{4}} \frac{h(z)}{(z-1)^3}\,dz$$

where $g(z) = \frac{e^{3z}}{(z-1)^3}$ and $h(z) = \frac{e^{3z}}{z}$. Since g is analytic on and inside of $|z| = \frac{1}{4}$ and h is analytic on and inside $|z-1| = \frac{1}{4}$, we can apply the Cauchy Integral Formulas to conclude that $I = 2\pi i \cdot g(0) + \frac{2\pi i}{2!} h''(1)$.

To Finish

Clearly, $g(0) = -1$. But

$$h'(z) = \frac{3e^{3z}z - e^{3z}}{z^2} = \frac{3e^{3z}}{z} - \frac{e^{3z}}{z^2}.$$

So

$$h''(z) = \frac{9e^{3z}z - 3e^{3z}}{z^2} - \frac{3e^{3z}z^2 - 2ze^{3z}}{z^4}.$$

Hence $h''(1) = \frac{e^3(9-3)}{1} - \frac{e^3(3-2)}{1} = 5e^3$. Hence

$$I = -2\pi i + \pi i 5e^3.$$

A Converse to Cauchy's Integral Theorem

Theorem (Moreara's Theorem)

Suppose that f is continuous on a domain D. If

$$\int_\Gamma f(z)\,dz = 0$$

for every closed contour Γ in D, then f is analytic in D.

Proof.

By our Antiderivative Theorem, f has an antiderivative F in D. Then F is analytic in D. But this implies $F' = f$ is also analytic. $\square$

Enough

Ok, that is the end of big day and a big week. Remember that this no class meeting on Monday next week. However, there there will be a pre-recorded lecture as usual.