# Math 43: Spring 2020
# Lecture 16 and 17 Summary

Dana P. Williams

Dartmouth College

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# Start ...

1. We should be recording.
2. We have the midterm coming up on Wednesday.
3. The midterm will cover through Monday's Lecture which will cover up to and including section 5.3 in the text.
4. I've put a copy of last year's midterm on the assignment page.

# Cauchy Again

**Theorem (Cauchy's Estimates)**

*Let $D = B_R(z_0)$. Suppose that $f$ is analytic on $D$ and bounded by $M$ on $D$; that is, that $|f(z)| \leq M$ for all $z \in D$. Then for each $n = 0, 1, 2, \ldots$, we have*

$$\left|f^{(n)}(z_0)\right| \leq \frac{n!M}{R^n}.$$

**Theorem (Liouville's Theorem)**

*A bounded entire function must be a constant function.*

**Theorem (Fundamental Theorem of Algebra)**

*Suppose that $p(z)$ is a polynomial of degree at least one, then $p(z)$ has at least one complex root.*

# The Maximum Modulus Principle

**Theorem (Maximum Modulus Principle)**

*Suppose that $f$ is analytic on a domain $D$. If there is a $z_0 \in D$ such that*

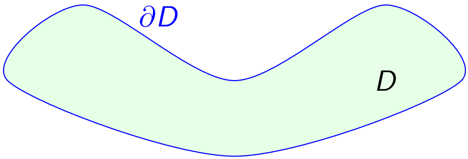
$$|f(z)| \le |f(z_0)| \quad \text{for all } z \in D,$$

*then $f$ is constant.*

**Remark**

1. Colloquially, a non-constant analytic function can not attain its maximum modulus on any domain.
2. What the Maximum Modulus Principle does not say: note that $f(z) = z$ is bounded by $M = 1$ on the domain $D = B_1(0) = \{ z : |z| < 1 \}$. Of course, it is not a constant function. That is because it does not attain its maximum modulus on $D$.

# The Extreme Value Theorem




**Remark (The Extreme Value Theorem from Calculus)**

If $D$ is a bounded domain, then the closure, $\overline{D}$, of $D$ is the union of $D$ and its boundary $\partial D$. Then $\overline{D}$ is closed and bounded. The the Extreme Value Theorem from multivariable calculus tells us that a continuous real-valued function $f$ on $\overline{D}$ must attain its maximum and minimum on $\overline{D}$. This means there are points $c, d \in \overline{D}$ such that $f(c) \leq f(z) \leq f(d)$ for all $z \in \overline{D}$. Since $\mathbf{C}$ is not ordered, none of this makes any sense at all for complex-valued functions!

# Maximum Modulus Principle for Bounded Domains

**Theorem**

*Suppose that $D$ is a bounded domain and that $f : \overline{D} \subset \mathbf{C} \to \mathbf{C}$ is a continuous function which is analytic in $D$. Then $f$ attains its maximum modulus on the boundary $\partial D$ of $D$. (In the text, the authors describe such a $f$ as analytic in $D$ and continuous up to and including the boundary.)*

**Remark (Minimum Modulus)**

Consider $f(z) = z$ on the domain $D = B_1(0) = \{ z : |z| < 1 \}$. Note that the minimum of $|f(z)|$ occurs at $0 \in D$. Of course, $f$ isn't a constant function. You will explore what can be said about a sort of "Minimum Modulus Principle" in the homework.

# Series

## Definition

A series of complex numbers is an expression of the form

$$\sum_{n=0}^{\infty} c_n \tag{$\dagger$}$$

where each $c_n \in \mathbf{C}$. The $n^{th}$-partial sum of ($\dagger$) is $s_n := c_0 + c_1 + \cdots + c_n$. If $\lim_{n\to\infty} s_n$ exists and equals $s$, then we say that ($\dagger$) converges with sum $s$. Then we write

$$s = \sum_{n=0}^{\infty} c_n.$$

Otherwise, we say that ($\dagger$) diverges.

# Geometric Series

## Definition

If $a \neq 0$ and $c \in \mathbf{C}$, then we say that

$$\sum_{n=0}^{\infty} ac^n = a + ac + ac^2 + \cdots$$

is a geometric series with ratio $c$.

## Theorem

*If $|c| < 1$, then*

$$\sum_{n=0}^{\infty} ac^n = \frac{a}{1-c}.$$

*If $|c| \geq 1$, then the series diverges.*

# Reverting to Calculus II

**Lemma**

*We have*

$$\sum_{n=0}^{\infty} c_n = s$$

*if and only if*

$$\sum_{n=0}^{\infty} \mathrm{Re}(c_n) = \mathrm{Re}(s) \quad \textit{and} \quad \sum_{n=0}^{\infty} \mathrm{Im}(c_n) = \mathrm{Im}(s).$$

**Proposition (The Comparison Test)**

*Suppose that* $|c_n| \leq a_n$ *for* $n \geq N$*. If* $\sum_{n=0}^{\infty} a_n < \infty$*, then* $\sum_{n=0}^{\infty} c_n$ *converges.*

# Absolute Convergence

## Definition

We say that a series $\sum_{n=0}^{\infty} c_n$ converges absolutely if $\sum_{n=0}^{\infty} |c_n| < \infty$.

## Proposition (Ratio Test)

*Suppose that*

$$\lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| = L$$

*exists. If $L < 1$, then $\sum_{n=0}^{\infty} c_n$ converges absolutely. If $L > 1$, then $\sum_{n=0}^{\infty} c_n$ diverges.*

# Pointwise vs. Uniform Convergence

## Definition

A sequence $(F_n)$ of functions on a set $D$ converges pointwise to a function $F$ on $D$ if $\lim_{n\to\infty} F_n(z) = F(z)$ for all $z \in D$. We say that series $\sum_{n=0}^{\infty} f_n(z)$ converges pointwise to $f(z)$ on $D$ if the partial sums $F_n(z) = \sum_{k=0}^{n} f_k(z)$ converge pointwise to $f$ on $D$.

## Definition

A sequence $(F_n)$ of functions on a set $D$ converges uniformly to a function $F$ on $D$ if for all $\epsilon > 0$ there is a $N = N(\epsilon)$ such that $n \geq N$ implies $|F(z) - F_n(z)| < \epsilon$ for all $z \in D$. We say that a series $\sum_{n=0}^{\infty} f_n(z)$ converges uniformly to $f$ on $D$, if the partial sums $F_n(z) = \sum_{k=0}^{n} f_k(z)$ converge uniformly to $f$ on $D$.

# An Example

## Example

Suppose that $z_0 \neq 0$.

1. The series $\sum_{n=0}^{\infty} \left(\frac{z}{z_0}\right)^n$ converges pointwise to $F(z) = z_0/(z_0 - z)$ on $D = B_{|z_0|}(0)$.
2. The convergence above is not uniform on $D$.
3. However, the convergence is uniform on $D_r = \overline{B_r(0)}$ whenever $0 < r < |z_0|$.