Math 43: Spring 2020
Lecture 16 Part I

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No Class Meeting Monday

There will be no regular class meeting on Monday, May 4th. Instead we will review the lecture and Wednesday's lecture during our class meeting on Wednesday, May 6th.

With no political agenda, but out of a sense of respect and decency, I would like to dedicate this lecture to the memory of those young students killed and to those injured 50 years ago at Kent State University on May 4, 1970.

Bounded Functions

Definition

We say that a function $f : D \subset \mathbf{C} \to \mathbf{C}$ is bounded on D if there is a $M > 0$ such that $|f(z)| \leq M$ for all $z \in D$. Alternatively, we say that f is bounded by M on D if $|f(z)| \leq M$ for all $z \in D$. If $D = \mathbf{C}$, then we just say that f is bounded or f is bounded by M.

Remark

The point of highlighting this pedantry is to emphasize that, since the complex numbers are not an ordered field, saying that f is bounded can only mean that the modulus of f is bounded.

Cauchy Again

Theorem (Cauchy's Estimates)

Let $D = B_R(z_0)$. Suppose that f is analytic on D and bounded by M on D; that is, that $|f(z)| \leq M$ for all $z \in D$. Then for each $n = 0, 1, 2, \ldots,$ we have

$$\left| f^{(n)}(z_0) \right| \leq \frac{n! M}{R^n}.$$

The Proof of Cauchy's Estimates

Proof.

Fix $0 < \rho < R$. Let C_ρ be the positively oriented circle of radius ρ centered at z_0. Since f is analytic on and inside of C_ρ, and z_0 since lies inside of C_ρ, Cauchy's Integral Formula for the Derivatives implies that

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{C_\rho} \frac{f(w)}{(w - z_0)^{n+1}}\, dw.$$

Then our usual estimates for a contour integral gives

$$|f^{(n)}(z_0)| \leq \frac{n!}{2\pi} \frac{M}{\rho^{n+1}} \cdot 2\pi\rho = \frac{n!M}{\rho^n}.$$

Therefore $|f^{(n)}(z_0)| \leq \lim_{\rho \nearrow R} \frac{n!M}{\rho^n} = \frac{n!M}{R^n}$. □

Liouville's Theorem

Now we come to one of the more celebrated results in the subject. As an added bonus, it doesn't carry Cauchy's name.

Theorem (Liouville's Theorem)

A bounded entire function must be a constant function.

Proof.

Suppose that f is entire and $|f(z)| \leq M$ for all $z \in \mathbf{C}$. Fix $z_0 \in \mathbf{C}$. Then for any $R > 0$, f is bounded by M in $B_R(z_0)$ and by Cauchy's Estimates,

$$|f'(z_0)| \leq \frac{M}{R}.$$

But, we we can take R as large as we please. Therefore $f'(z_0) = 0$. Since z_0 was arbitrary, we have $f'(z) = 0$ for all $z \in \mathbf{C}$ and hence f is constant. □

The Proof

Remark

So why is Liouville's Theorem so celebrated? Well, for lots of reasons, but one good one is that it allows us to give a very straightforward proof of the Fundamental Theorem of Algebra.

Fundamental Theorem of Algebra

Theorem (Fundamental Theorem of Algebra)

Suppose that $p(z)$ is a polynomial of degree at least one, then $p(z)$ has at least one complex root.

Proof.

Suppose that $\deg p(z) = n \geq 1$ and $p(z)$ is never zero. By a homework problem, there is a $R > 0$ such that $|z| \geq R$ implies,

$$|p(z)| \geq K|z|^n \geq KR^n.$$

Proof continued

Proof Continued.

Since we're assuming $p(z)$ is never zero, $h(z) = \frac{1}{p(z)}$ is an entire function. Since $z \mapsto |h(z)|$ is continuous on the closed bounded set $\{ z : |z| \leq R \}$, there is a M_0 such that $|h(z)| \leq M_0$ if $|z| \leq R$. On the other hand, if $|z| \geq R$, we have

$$|h(z)| = \left| \frac{1}{p(z)} \right| \leq \frac{1}{KR^n}.$$

This means that for all $z \in \mathbf{C}$,

$$|h(z)| \leq \max\{M_0, \frac{1}{KR^n}\}.$$

Therefore h is bounded and constant by Liouville's Theorem. But $p(z)$ is not constant since $\deg p(z) \geq 1$. This is a contradiction and completes the proof. □

Proposition

Suppose that u is harmonic on $\mathbf{R}^2$. Suppose that u is bounded above by M. That is, assume $u(x,y) \leq M$ for all $(x,y) \in \mathbf{R}^2$. Then u is constant.

Proof.

Since $\mathbf{C}$ is simply connected, u has a harmonic conjugate v. Thus $f(z) = u(z) + iv(z)$ is entire. Let $g(z) = \exp(f(z)) = e^{f(z)}$. Then g is entire. Then since the function $x \mapsto e^x$ is strictly increasing on $\mathbf{R}$,

$$|g(z)| = |e^{f(z)}| = |e^{u(z)}e^{iv(z)}| = e^{u(z)} \leq e^M.$$

Therefore g is constant by Liouville's Theorem. Then for all z, $0 = g'(z) = f'(z)e^{g(z)}$. Since $e^{g(z)}$ never vanishes, $f'(z) = 0$ for all z and f is constant. Therefore u is constant. $\square$

Break Time

Time for a break. We have another cool result coming up in the second part of the lecture.