# Math 43: Spring 2020
# Lecture 16 Part II

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**Example**

The function $u(x,y) = 2 - \dfrac{1}{1-x^2-y^2}$ is well-defined on the unit disk $D = B_1(0) = \{\,(x,y) : x^2+y^2 < 1\,\}$. While $u$ is not bounded below, it does have a maximum value of 1 at the point $(0,0) \in D$. We will see now that $u$ is not of the form $u(z) = |f(z)|$ with $f$ analytic in $D$.

## Theorem (Maximum Modulus Principle)

*Suppose that $f$ is analytic on a domain $D$. If there is a $z_0 \in D$ such that*

$$|f(z)| \leq |f(z_0)| \quad \text{for all } z \in D,$$

*then $f$ is constant.*

## Remark

1. Colloquially, a non-constant analytic function can not attain its maximum modulus on any domain.
2. The homework problem that an analytic function with constant modulus on a domain $D$ must be constant is a special case.
3. What the Maximum Modulus Principle does not say: note that $f(z) = z$ is bounded by $M = 1$ on the domain $D = B_1(0) = \{ z : |z| < 1 \}$. Of course, it is not a constant function. That is because it does not attain its maximum modulus on $D$.

# Lemma A

## Lemma (A)

*Suppose* $f : [0, 2\pi] \to \mathbf{R}$ *is continuous and such that* $0 \leq f(x) \leq M$ *for all* $x \in [0, 2\pi]$*. If*

$$M = \frac{1}{2\pi} \int_0^{2\pi} f(x)\, dx,$$

*then* $f(x) = M$ *for all* $x \in [0, 2\pi]$*.*

## Proof.

We have

$$0 = \frac{1}{2\pi} \int_0^{2\pi} \bigl(M - f(x)\bigr)\, dx.$$

Since $M - f(x) \geq 0$ and $x \mapsto M - f(x)$ is continuous, we must have $M - f(x) = 0$ for all $x$. $\square$

# Lemma B

## Lemma (A)

*Suppose that $f$ is analytic in the disk $D = B_R(z_0)$ and that*

$$|f(z)| \leq |f(z_0)| \quad \textit{for all } z \in D.$$

*Then $f$ is constant.*

## Proof.

You showed on homework (§4.5 #8) that the Cauchy Integral Formula implies that for all $0 < r < R$, we have

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta.$$

# Proof Continued

**Proof.**

Let $M = |f(z_0)|$. Then

$$M = |f(z_0)| \le \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + re^{i\theta})|\,d\theta \le \frac{1}{2\pi} M \cdot 2\pi = M.$$

Hence

$$M = \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + re^{i\theta})|\,d\theta.$$

Therefore by Lemma A, $M = |f(z_0 + re^{i\theta})|$ for all $\theta \in [0, 2\pi]$.

# Proof Continued

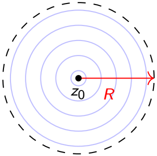



Figure: The modulus is constant on every intermediate circle

**Proof.**

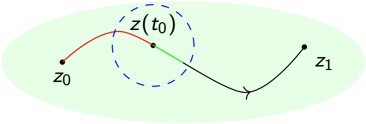
Since the modulus is constantly equal to $M = |f(z_0)|$ on every circle of radius $0 < r < R$, the modulus is constant in all of $D$. But we proved on homework that this implies $f$ itself is constant. This proves the lemma. □

# Proof of the MMP

## Proof of the Theorem.

Suppose that for $z_0 \in D$, we have $|f(z)| \leq |f(z_0)|$ for all $z \in D$. To see that this forces $f$ to be constant, it will suffice to see that $|f(z)|$ is constant on $D$. Suppose to the contrary of what we want to prove, $|f(z)|$ is not constant. Then there is a $z_1 \in D$ such that $|f(z_1)| < |f(z_0)|$. Since $D$ is connected, there is a contour $\Gamma$ in $D$ from $z_0$ to $z_1$. Let $z : [0, 1] \to D$ be an admissible parameterization of $\Gamma$. Note that $z(0) = z_0$ and $z(1) = z_1$. $\square$

# Proof of MMP

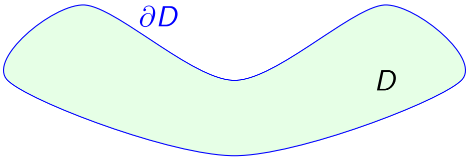



Let $A = \{ s \in [0,1] : |f(z(t))| = |f(z_0)| \text{ for all } 0 \le t \le s \}$. Let $t_0 = \text{l.u.b.}\, A$. Note that $0 \in A$ and $1 \notin A$.

Since $|f(z(t))| = |f(z_0)|$ for all $0 \le t < t_0$, we must have $t_0 \in A$ by continuity. Moreover, if $t_0 < t_1$, then there is $t_0 < t < t_1$ such that $|f(z(t))| < |f(z_0)|$. But there is a $r > 0$ such that $B_r(z(t_0)) \subset D$. But then for all $z \in B_r(z(t_0))$, we have $|f(z)| \le |f(z(t_0))|$. Then by Lemma B, $f$ is constant in $B_r(z(t_0))$. This contradicts our choice of $t_0$.

This completes the proof of the Maximum Modulus Principle.

# The Extreme Value Theorem




**Remark (The Extreme Value Theorem from Calculus)**

If $D$ is a bounded domain, then the closure, $\overline{D}$, of $D$ is the union of $D$ and its boundary $\partial D$. Then $\overline{D}$ is closed and bounded. The the Extreme Value Theorem from multivariable calculus tells us that a continuous real-valued function $f$ on $\overline{D}$ must attain its maximum and minimum on $\overline{D}$. This means there are points $c, d \in \overline{D}$ such that $f(c) \leq f(z) \leq f(d)$ for all $z \in \overline{D}$. Since $\mathbf{C}$ is not ordered, none of this makes any sense at all for complex-valued functions!

# Maximum Modulus Principle for Bounded Domains

## Theorem

*Suppose that $D$ is a bounded domain and that $f : \overline{D} \subset \mathbf{C} \to \mathbf{C}$ is a continuous function which is analytic in $D$. Then $f$ attains its maximum modulus on the boundary $\partial D$ of $D$. (In the text, the authors describe such a $f$ as analytic in $D$ and continuous up to and including the boundary.)*

## Proof.

The real-valued function $z \mapsto |f(z)|$ must attain its maximum on $\overline{D}$. If this occurs on $\partial D$, then we have nothing to show. However, if the maximum is attained in the interior, namely in $D$, then the Maximum modulus principle implies that $f$ is constant on $D$. Then by continuity, $f$ is constant on $\overline{D}$. Therefore by default the maximum of $|f(z)|$ must also occur on $\partial D$. $\square$

# Minimums

**Remark**

Consider $f(z) = z$ on the domain $D = B_1(0) = \{ z : |z| < 1 \}$. Note that the minimum of $|f(z)|$ occurs at $0 \in D$. Of course, $f$ isn't a constant function. You will explore what can be said about a sort of "Minimum Modulus Principle" in the homework.

**Example**

Suppose that $u$ is a harmonic function on a simply connected domain $D$. Suppose that there is a point $(x_0, y_0) \in D$ such that $u(x, y) \leq u(x_0, y_0)$ for all $(x, y) \in D$. In other words, assume $u$ attains its maximum on $D$. Show that $u$ must be constant.

# Solution

## Solution.

Let $z_0 = x_0 + iy_0$. Since $D$ is simply connected, there is a harmonic conjugate $v$ for $u$ in $D$. Then $f(z) = u(z) + iv(z)$ is analytic in $D$. Hence $g(z) = \exp(f(z))$ is analytic in $D$. But then since $x \mapsto e^x$ is strictly increasing on $\mathbf{R}$, we have

$$|g(z)| = |e^{f(z)}| = e^{u(z)} \le e^{u(z_0)} \quad \text{for all } z \in D.$$

Therefore $g$ is constant by the Maximum Modulus Principle. Therefore for all $z \in D$, $0 = g'(z) = f'(z)e^{f(z)}$. Since $e^{f(z)}$ never vanishes, we must have $f'(z) = 0$ for all $z \in D$. Thus $f$ is constant, and hence $u$ is constant. □

# Enough

That is enough for one lecture.