Math 43: Spring 2020
Lecture 17 Part I

Dana P. Williams

Dartmouth College

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Definition

A series of complex numbers is an expression of the form

$$\sum_{n=0}^{\infty} c_n \tag{†}$$

where each $c_n \in \mathbf{C}$. The n^{th}-partial sum of (†) is $s_n := c_0 + c_1 + \cdots + c_n$. If $\lim_{n\to\infty} s_n$ exists and equals s, then we say that (†) converges with sum s. Then we write

$$s = \sum_{n=0}^{\infty} c_n.$$

Otherwise, we say that (†) diverges.

A Admonition

Remark

If you remember nothing else from this lecture, please keep in mind that the "sum of a series" is not a sum. It is the limit of the sequence of partial sums.

Now we come to the king of series. It is one of the few that we can work with directly, and it is fundamental for what is to come.

Definition

If $a \neq 0$ and $c \in \mathbf{C}$, then we say that

$$\sum_{n=0}^{\infty} ac^n = a + ac + ac^2 + \cdots$$

is a geometric series with ratio c.

Geometric Series

Theorem

If $|c| < 1$, *then*

$$\sum_{n=0}^{\infty} ac^n = \frac{a}{1-c}.$$

If $|c| \geq 1$, *then the series diverges.*

Proof.

If $c \neq 1$, then as we have used repeatedly, $s_n = a + ac + \cdots ac^n = a(1 + c + c^2 + \cdots + c^n) = a\frac{1-c^{n+1}}{1-c}$. But if $|c| < 1$, then $\lim_{n\to\infty} |c^n| = \lim_{n\to\infty} |c|^n = 0$. Hence if $|c| < 1$, then

$$\lim_{n\to\infty} s_n = a\frac{1-0}{1-c} = \frac{a}{1-c}.$$

This proves the first assertion. The second is a homework problem. $\square$

Lemma

We have

$$\sum_{n=0}^{\infty} c_n = s$$

if and only if

$$\sum_{n=0}^{\infty} \operatorname{Re}(c_n) = \operatorname{Re}(s) \quad \textit{and} \quad \sum_{n=0}^{\infty} \operatorname{Im}(c_n) = \operatorname{Im}(s).$$

Proof.

We have $\lim_n s_n = s$ if and only if $\lim_n \operatorname{Re}(s_n) = \operatorname{Re}(s)$ and $\lim_n \operatorname{Im}(s_n) = \operatorname{Im}(s)$. $\square$

The Comparison Test

Proposition

Suppose that $|c_n| \leq a_n$ *for* $n \geq N$*. If* $\displaystyle\sum_{n=0}^{\infty} a_n < \infty$*, then* $\displaystyle\sum_{n=0}^{\infty} c_n$ *converges.*

Proof.

Suppose that $\sum_{n=0}^{\infty} a_n < \infty$. Since we have $|\operatorname{Re}(c_n)| \leq |c_n| \leq a_n$, $\sum_{n=0}^{\infty} \operatorname{Re}(c_n)$ converges by the Calculus II comparison test. Similarly, $\sum_{n=0}^{\infty} \operatorname{Im}(c_n)$ must converge. Now apply the previous result. $\square$

Absolute Convergence

Definition

We say that a series $\sum_{n=0}^{\infty} c_n$ converges absolutely if $\sum_{n=0}^{\infty} |c_n| < \infty$.

Remark

It follows from the comparison test that an absolutely convergent series is convergent. We know from Calculus II that the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \cdots$$

is convergent by not absolutely convergent.

Ratio Test

Proposition (Ratio Test)

Suppose that

$$\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = L$$

exists. If $L < 1$, then $\sum_{n=0}^{\infty} c_n$ converges absolutely. If $L > 1$, then $\sum_{n=0}^{\infty} c_n$ diverges.

Remark

The proof goes by applying the Calculus II ratio test to the real and imaginary parts. Note that if $L = 1$, then the test gives no information. The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ with $p > 0$ converges if $p > 1$ and diverges if $p \leq 1$. But the ratio test yields that value $L = 1$ in all cases. (Such a series was called a p-series back in Calculus II.)

Remark

Infinite series are here to stay in Math 43. Some review on your part would not go astray.

Time for a Break.