Math 43: Spring 2020
Lecture 17 Part II

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An Important Example

If $z_0 \neq 0$ and $|z| < |z_0|$, then since $\left|\frac{z}{z_0}\right| < 1$, we can sum a geometric series to get a function on $D = B_{|z_0|}(0)$ given by

$$F(z) = \sum_{n=0}^{\infty} \left(\frac{z}{z_0}\right)^n = 1 + \frac{z}{z_0} + \left(\frac{z}{z_0}\right)^2 + \cdots = \frac{1}{1 - \frac{z}{z_0}} = \frac{z_0}{z_0 - z}.$$

This just means that provided $|z| < |z_0|$, the n^{th}-partial sum

$$F_n(z) = \sum_{k=0}^{n} \left(\frac{z}{z_0}\right)^k$$

satisfies

$$\lim_{n \to \infty} F_n(z) = F(z) = \frac{z_0}{z_0 - z}.$$

Pointwise Convergence

Definition

A sequence (F_n) of functions on a set D converges pointwise to a function F on D if $\lim_{n\to\infty} F_n(z) = F(z)$ for all $z \in D$. We say that series $\sum_{n=0}^{\infty} f_n(z)$ converges pointwise to $f(z)$ on D if the partial sums $F_n(z) = \sum_{k=0}^{n} f_k(z)$ converge pointwise to f on D.

Thus in our example, $D = B_{|z_0|}(0)$. This means that for all $\epsilon > 0$ there is a $N = N(\epsilon, z)$ such that $n \geq N$ implies $|F(z) - F_n(z)| < \epsilon$. That is

$$\left|\frac{z_0}{z_0 - z} - \sum_{k=0}^{n}\Big(\frac{z}{z_0}\Big)^k\right| = \left|\frac{1}{1-\frac{z}{z_0}} - \left(\frac{1-\big(\frac{z}{z_0}\big)^{n+1}}{1-\frac{z}{z_0}}\right)\right|$$
$$= \left|\frac{\big(\frac{z}{z_0}\big)^{n+1}}{1-\frac{z}{z_0}}\right| < \epsilon$$

provided $n \geq N = N(\epsilon, z)$.

Question

Saying that $N = N(\epsilon, z)$ means I get to know what both ϵ and z are in order to figure out how large N has to be. But what if I wanted an N that only depended on ϵ and not which $z \in D$ we pick? Let $z_n := \left(\frac{1}{2}\right)^{\frac{1}{n+1}} z_0$. Then $|z_n| < |z_0|$ so that $z_n \in D = B_{|z_0|}(0)$. But then

$$|F(z_n) - F_n(z_n)| = \left| \frac{\left(\frac{z_n}{z_0}\right)^{n+1}}{1 - \frac{z_n}{z_0}} \right|$$
$$= \frac{\frac{1}{2}}{1 - \left(\frac{1}{2}\right)^{\frac{1}{n+1}}} > \frac{1}{2}$$

All this just means that if $\epsilon = \frac{1}{2}$, then there is no single N depending only on ϵ such that $n \geq N$ implies

$$|F(z) - F_n(z)| < \epsilon \quad \text{for all } z \in D = B_{|z_0|}(0).$$

Uniform Convergence

Definition

A sequence (F_n) of functions on a set D converges uniformly to a function F on D if for all $\epsilon > 0$ there is a $N = N(\epsilon)$ such that $n \geq N$ implies $|F(z) - F_n(z)| < \epsilon$ for all $z \in D$. We say that a series $\sum_{n=0}^{\infty} f_n(z)$ converges uniformly to f on D, if the partial sums $F_n(z) = \sum_{k=0}^{n} f_k(z)$ converge uniformly to f on D.

Remark

In our example, $\displaystyle\sum_{n=0}^{\infty}\Big(\frac{z}{z_0}\Big)^n$ converges pointwise to $F(z) = \frac{z_0}{z_0 - z}$ on $D = B_{|z_0|}(0)$, but we showed that it does not converge uniformly on D.

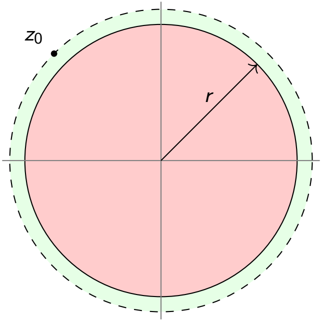
Almost!

Proposition

The series

$$\sum_{n=0}^{\infty}\left(\frac{z}{z_0}\right)^n = \frac{1}{1-\frac{z}{z_0}}$$

converges uniformly on $D_r = \overline{B_r(0)}$ *whenever* $0 < r < |z_0|$.



Proof.

We already computed that

$$|F(z) - F_n(z)| = \left| \frac{\left(\frac{z}{z_0}\right)^{n+1}}{1 - \frac{z}{z_0}} \right| \le \frac{\left|\frac{z}{z_0}\right|^{n+1}}{1 - \left|\frac{z}{z_0}\right|}$$

Then if $z \in D_r = \overline{B_r(0)}$, then $|z| \le r$ and

$$|F(z_n) - F_n(z_n)| \le \frac{\left(\frac{r}{|z_0|}\right)^{n+1}}{1 - \frac{r}{|z_0|}}. \tag{‡}$$

Since $|\frac{r}{|z_0|}| < 1$, $\left(\frac{r}{|z_0|}\right)^{n+1} \to 0$. Since the right-hand side of (‡) does not depend on z, there is a $N = N(\epsilon)$ such that $n \ge N$ implies

$$\frac{\left(\frac{r}{|z_0|}\right)^{n+1}}{1 - \frac{r}{|z_0|}} < \epsilon.$$

This what we needed. □

Break Time

Ok, so now we know enough about series, pointwise convergence, and uniform convergence. Next time, back to analytic functions! For that we will need to introduce Taylor series.

Enough for now.