# Math 43: Spring 2020
# Lecture 18 Summary

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# This Week

1. We should be recording.
2. We will have our midterm on Wednesday.
3. The midterm will cover up to and including today's lecture.
4. This means up to and including §5.3 of the text.
5. I will distribute the exam after class time on Wednesday, and you must upload your completed exam by the beginning of class on Friday.

# Taylor Series

**Definition**

Suppose that $f$ is analytic at $z_0$. Then the Taylor series for $f$ about $z_0$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n.$$

When $z_0 = 0$, this series is also called the MacLaurin series for $f$.

**Theorem (Taylor's Theorem)**

*Suppose that $f$ is analytic in $B_R(z_0)$. Then the Taylor series for $f$ about $z_0$ converges absolutely to $f(z)$ for all $z \in B_R(z_0)$. Furthermore the convergence if uniform in every closed sub-disk*

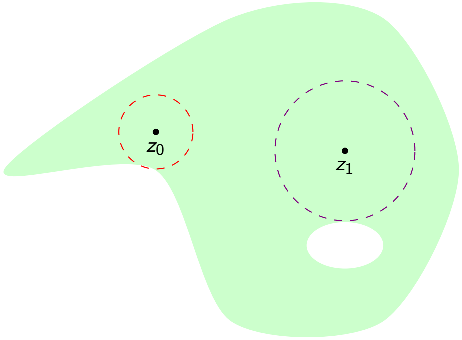
$$\overline{B_r(z_0)} = \{ z \in \mathbf{C} : |z - z_0| \leq r \}$$

*provided only that $0 < r < R$.*

# Cool

## Remark

Taylor's Theorem is way cooler than we have any right to expect based on our Calculus II experience. If $f$ is analytic in a domain $D$ and $z_0 \in D$, then the Taylor series for $f$ converges to $f(z)$ in the largest disk $B_R(z_0)$ such that $B_R(z_0) \subset D$!



### Theorem

*Suppose that $f$ is analytic in a disk $B_R(z_0)$ with Taylor series*

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n \quad \textit{for all } |z - z_0| < R.$$

*Then the Taylor series for $f'$ is given by "term-by-term differentiation". That is,*

$$f'(z) = \sum_{n=1}^{\infty} n a_n(z - z_0)^{n-1} \quad \textit{for all } |z - z_0| < R.$$

## Definition

The Cauchy product of the series

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n \quad \text{and} \quad \sum_{n=0}^{\infty} b_n(z-z_0)^n$$

is given by

$$\sum_{n=0}^{\infty} c_n(z-z_0)^n$$

where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

# More Automatic Miracles

## Theorem

*Suppose that $f$ and $g$ are analtyic at $z_0$ with Taylor series*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n$$

*about $z_0$. Then the Taylor series for $fg$ about $z_0$ is given by the Cauchy product*

$$(fg)(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

*where*

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$