Math 43: Spring 2020
Lecture 18 Part I

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Friday May 8, 2020

Taylor Series

Definition

Suppose that f is analytic at z_0. Then the Taylor series for f about z_0 is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n.$$

When $z_0 = 0$, this series is also called the MacLaurin series for f.

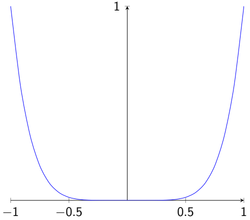
Example

If $f(z) = e^z$, then the MacLaurin series for f is the usual one:

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \cdots .$$

The Ratio test tells us that this series converges for all $z \in \mathbf{C}$. Of course, if z is real, we know this series converges to e^z. But what if z is complex? Nothing says that the sum, call it $f(z)$, has to equal e^z if $z \notin \mathbf{R}$.

In the Real World



Example

Consider the function $f : \mathbf{R} \to \mathbf{R}$ given by

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0, \text{ and} \\ 0 & \text{if } x = 0. \end{cases}$$

For fun, you can easily check that $f'(0)$ exists and is equal to 0.

If you are clever, you can use an induction proof to show that $f^{(n)}(0)$ exists and is equal to 0 for all $n = 0, 1, 2, \dots$. Thus f is smooth, and has a MacLaurin series. However, every term in the series is zero, so the MacLaurin series for f only converges to f at $x = 0$. Thus in the "real world", a smooth function need not be given by its Taylor Series. Even worse, we can construct functions whose Taylor series converges only at z_0!?!

A Complex Miracle

Theorem (Taylor's Theorem)

Suppose that f is analytic in $B_R(z_0)$. Then the Taylor series for f about z_0 converges absolutely to $f(z)$ for all $z \in B_R(z_0)$. Furthermore the convergence if uniform in every closed sub-disk

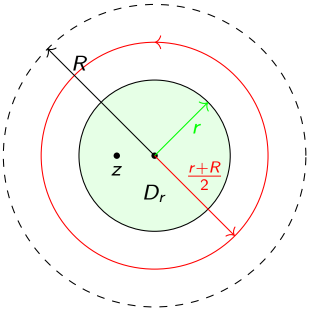
$$\overline{B_r(z_0)} = \{ z \in \mathbf{C} : |z - z_0| \leq r \}$$

provided only that $0 < r < R$.

Remark

This is cool and totally amazing given what can go wrong in the real case. While the proof is far from easy, it is not really that bad given the Cauchy Integral Formula and what we know about geometric series.

The Proof



Fix $0 < r < R$. Let $D_r = \overline{B_r(z_0)}$. It will suffice to prove that

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

converges uniformly to $f(z)$ on D_r.

Let C be the positively oriented circle $|z - z_0| = \frac{r+R}{2}$. Then if $z \in D_r$, we have

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z}\, dz.$$

Getting Geometric

Since $\dfrac{1}{1-c} = 1 + c + c^2 + \cdots + c^n + \dfrac{c^{n+1}}{1-c}$, we have

$$\frac{1}{w-z} = \frac{1}{(w-z_0)-(z-z_0)} = \frac{1}{w-z_0}\cdot\frac{1}{1-\frac{z-z_0}{w-z_0}}$$

$$= \frac{1}{w-z_0}\cdot\left[1+\frac{z-z_0}{w-z_0}+\cdots\left(\frac{z-z_0}{w-z_0}\right)^n+\frac{\left(\frac{z-z_0}{w-z_0}\right)^{n+1}}{1-\frac{z-z_0}{w-z_0}}\right]$$

$$= \frac{1}{w-z_0}+\frac{z-z_0}{(w-z_0)^2}+\cdots+\frac{(z-z_0)^n}{(w-z_0)^{n+1}} + \frac{(z-z_0)^{n+1}}{(w-z)(w-z_0)^{n+1}}$$

Return

Plugging into CIF

Since $f(z) = \dfrac{1}{2\pi i} \displaystyle\int_C f(w) \cdot \dfrac{1}{w - z} \, dz$, we have

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z_0} \, dw + \cdots + \frac{(z - z_0)^n}{2\pi i} \int_C \frac{f(w)}{(w - z_0)^{n+1}} \, dw + T_n(z)$$

where

$$T_n(z) = \frac{(z - z_0)^{n+1}}{2\pi i} \int_C \frac{f(w)}{(w - z)(w - z_0)^{n+1}} \, dw.$$

Therefore, by CIF for the derivatives,

$$f(z) = \underbrace{f(z_0) + f'(z_0)(z - z_0) + \cdots + \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n}_{F_n(z)} + T_n(z).$$

Suffices

Since $F_n(z)$ is the n^{th}-partial sum for the Taylor series of f, it will suffice to see that given $\epsilon > 0$ there is a $N = N(\epsilon)$ such that $|T_n(z)| < \epsilon$ if $z \in D_r$.
But if $z \in D_r$ and $w \in C$, then

$$|w - z_0| = \frac{r + R}{2}.$$

Also

$$\left|\frac{z - z_0}{w - z_0}\right| \leq \frac{r}{\frac{r+R}{2}} = \frac{2r}{R + r} < 1.$$

Moreover,

$$|w - z| \geq |w - z_0| - |z - z_0| \geq \frac{r + R}{2} - r = \frac{R - r}{2} > 0.$$

Finish

Finishing the Proof.

Thus if $|f(w)| \leq M$ for all $w \in C$, and if $r \in D_r$, then

$$
\begin{aligned}
|T_n(z)| &\leq \frac{1}{2\pi}\Big|\int_C \frac{f(w)(z-z_0)^{n+1}}{(w-z)(w-z_0)^{n+1}}\,dw\Big| \\
&\leq \frac{1}{2\pi}\frac{M}{\frac{R-r}{2}}\Big(\frac{2r}{R+r}\Big)^{n+1} 2\pi \frac{R+r}{2} \\
&\leq \frac{1}{2\pi} M \cdot \frac{R+r}{R-r}\Big(\frac{2r}{R+r}\Big)^{n+1}.
\end{aligned}
$$

Since $\big|\frac{2r}{R+r}\big| < 1$, $\big(\frac{2r}{R+r}\big)^{n+1} \to 0$. Hence there is a $N = N(\epsilon)$ such that $n \geq N$ implies $|T_n(z)| < \epsilon$ provided $z \in D_r$. This is what we wanted to prove. □

The Complex Exponential Function

Example

If $f(z) = e^z$, then $f^{(n)}(z) = e^z$ and hence $f^{(n)}(0) = 1$ for $n = 0, 1, 2, \ldots$. Hence the MacLaurin series for $f(z) = e^z$ is, as we said earlier, given by the usual one. Since e^z is entire, it is analytic in every disk of the form $B_R(0)$. Hence

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \cdots \quad \text{for all } z \in \mathbf{C}.$$

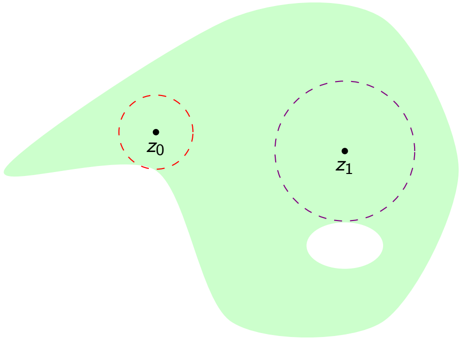
Remark

This certainly fully justifies our definition of the complex exponential function from back in week 1.

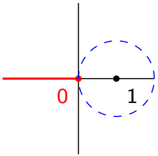
Cool

Remark

Taylor's Theorem is way cooler than we have any right to expect based on our Calculus II experience. If f is analytic in a domain D and $z_0 \in D$, then the Taylor series for f converges to $f(z)$ in the largest disk $B_R(z_0)$ such that $B_R(z_0) \subset D$!



The Principal Branch $\text{Log}(z)$



Let $f(z) = \text{Log}(z)$ and let $z_0 = 1$. Then f is analytic in $B_1(1)$. Therefore we know from the start that

$$\text{Log}(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(z-1)^n \qquad (\dagger)$$

for all $|z-1| < 1$.

But $f^{(0)}(z) = f(z) = \text{Log}(z)$, so $f(1) = 0$. But $f'(z) = \frac{1}{z}$, so $f'(1) = 1$. I leave it to you to check that $f^{(n)}(z) = (-1)^{n-1}\frac{(n-1)!}{z^n}$. Hence in general, $f^{(n)}(1) = (-1)^{n-1}(n-1)!$. Plugging into $(\dagger)$,

$$\text{Log}(z) = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{(z-1)^n}{n} = (z-1) - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} - \cdots$$

for all $|z-1| < 1$.

More Good Stuff to Come

Remark

We investigate some more miraculous properties of Taylor series for analytic functions in the second part of the lecture.

Time for a Break.