Math 43: Spring 2020
Lecture 18 Part II

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Derivatives

Theorem

Suppose that f is analytic in a disk $B_R(z_0)$ with Taylor series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{for all } |z - z_0| < R.$$

Then the Taylor series for f' is given by "term-by-term differentiation". That is,

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1} \quad \text{for all } |z - z_0| < R.$$

The Proof

Proof.

By definition

$$a_n = \frac{f^{(n)}(z_0)}{n!}.$$

On the other hand, f' is analytic in $B_R(z_0)$ and by Taylor's Theorem

$$f'(z) = \sum_{n=0}^{\infty} b_n(z - z_0)^n = \sum_{n=1}^{\infty} b_{n-1}(z - z_0)^{n-1}$$

where

$$b_{n-1} = \frac{(f')^{(n-1)}(z_0)}{(n-1)!} = \frac{f^{(n)}(z_0)}{n!} \cdot n = n \cdot a_n. \qquad \square$$

Example

Example

For $z \in B_1(1)$,

$$\operatorname{Log}(z) = (z-1) - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} - \cdots,$$

so if $|z-1| < 1$, we have

$$\begin{aligned} \frac{d}{dz}\big(\operatorname{Log}(z)\big) &= 1 - (z-1) + (z-1)^2 - \cdots \\ &= \frac{1}{1-(-(z-1))} \\ &= \frac{1}{z}. \end{aligned}$$

A Question

Remark

What should it mean to take the product of two Taylor series? Well, naively, we could just treat them as big polynomials:

$$(a_0 + a_1 z + a_2 z^2 + \cdots)(b_0 + b_1 z + b_2 z^2 + \cdots) \\ = a_0 b_0 + (a_1 b_0 + a_0 b_1) z + \cdots .$$

So the question that arises is to what extent does this make sense? Does the product series even converge? Is it the Taylor series of the product? In order to answer this, we have to get precise about the product formula.

The Cauchy Product

Definition

The Cauchy product of the series

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n \quad \text{and} \quad \sum_{n=0}^{\infty} b_n(z-z_0)^n$$

is given by

$$\sum_{n=0}^{\infty} c_n(z-z_0)^n$$

where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

Calculus is Always Good

Theorem (Leibniz Formula)

Suppose that f and g have derivatives of all orders at z_0. Then

$$(fg)^{(n)}(z_0) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(z_0) g^{(n-k)}(z_0)$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Proof.

This is a homework problem. □

Example

$(fg)'''(z) = f'''(z)g(z) + 3f''(z)g'(z) + 3f'(z)g''(z) + f(z)g'''(z).$

More Automatic Miracles

Theorem

Suppose that f and g are analtyic at z_0 with Taylor series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n$$

about z_0. Then the Taylor series for fg about z_0 is given by the Cauchy product

$$(fg)(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

The Proof

Proof.

Let $\sum_{n=0}^{\infty} d_n(z-z_0)^n$ be the Taylor series for fg about z_0. Then

$$\begin{aligned} d_n &= \frac{(fg)^{(n)}(z_0)}{n!} \\ &= \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k} f^{(k)}(z_0)g^{(n-k)}(z_0) \\ &= \frac{1}{n!}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!} f^{(k)}(z_0)g^{(n-k)}(z_0) \\ &= \sum_{k=0}^{n}\frac{f^{(k)}(z_0)}{k!}\cdot\frac{g^{(n-k)}(z_0)}{(n-k)!} \\ &= \sum_{k=0}^{n} a_k b_{n-k} = c_n. \end{aligned}$$

$\square$

Example

Find the first few terms of the MacLaurin series for $f(z) = \frac{1}{1+e^z}$.

Solution

Repeated differentiation is not the way to go here! If you don't believe me, try finding $f^{(4)}(z)$! Instead, let $f(z) = a_0 + a_1 z + \cdots$. Then we have

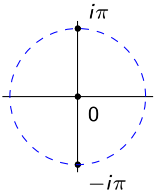
$$1 = (1+e^z)f(z) = (2 + z + \frac{z^2}{2} + \cdots)(a_0 + a_1 z + \cdots)$$
$$= 2a_0 + (2a_1 + a_0)z + (2a_2 + a_1 + \frac{a_0}{2})z^2 + \cdots$$

Hence $2a_0 = 1$ and $a_0 = \frac{1}{2}$. Similarly, $2a_1 + a_0 = 0$, and $a_1 = -\frac{1}{4}$. Then you can easily see that $a_2 = 0$, and keep going if you wish.

If you are bored, or have access to Mathematica or some such, then you can verify that

$$f(z) = \frac{1}{2} - \frac{z}{4} + \frac{z^3}{48} - \frac{z^5}{480} + \frac{17z^7}{80640} - \frac{31z^9}{1451520} + \cdots.$$

Although you compute as many coefficients as you like, there is no known general formula for the n^{th}-term. Nevertheless, we know what the radius of convergence is! The series has to converge out to the first place that f fails to be analytic. In this case, $e^z = -1$ when $z = i\pi + 2\pi ki$. Hence the radius of converges is at least π. Shortly, we'll see it can't be any larger.



More Good Stuff to Come

Remark

Taylor's Theorem is a very powerful result. We'll have a lot more to say about its consequences for analytic functions.

Enough for Today.