Math 43: Spring 2020
Lecture 19 Summary

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This Week

1. We should be recording. Also, "extra credit" points for anyone who reminds me to stop recording when this review is over.
2. Our Exam Wednesday covers through and including section 5.3 in the text and all of lecture 19 (aka this one).
3. There is a practice exam as well as solutions to the homework up to and including today's assignment available on the assignments page.
4. If at all possible, your neat and polished solutions, should be written on the exam itself—with extra pages added if necessary—including the cover page (with your name) and score sheet.
5. Your solutions should be uploaded by class time on Friday (10:10am EDT).
6. Keep in mind that you can bring me questions both in office hours tomorrow and during or class meeting time on Wednesday.
7. Just as with the preliminary exam, Friday's lecture will be recorded during our class meeting time.

Power Series

Definition

A series of the form

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n \tag{1}$$

with $a_n, z_0 \in \mathbf{C}$ is called a power series centered at z_0.

Theorem

Let $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ be a power series centered at z_0. Then there is a $0 \leq R \leq \infty$ such that

1. *the series converges absolutely if $|z - z_0| < R$,*
2. *The series diverges if $|z - z_0| > R$, and*
3. *the series converges uniformly on any closed subdisk $D_r = \{ z : |z - z_0| \leq r \}$ provided $0 < r < R$.*

A Picture to Remember

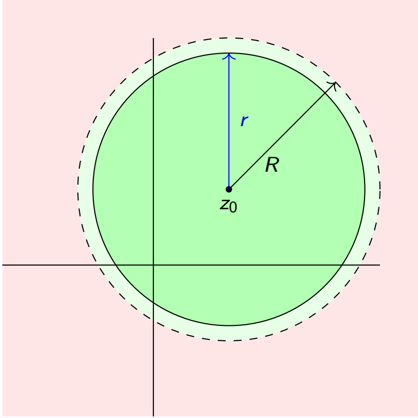

Figure: The Radius of Convergence of a power series $\sum_{n=0}^{\infty} a_n(z - z_0)$

Uniform is Good

Theorem

Suppose that (f_n) is a sequence of continuous complex-valued functions on a set $D \subset \mathbf{C}$. If $f_n \to f$ uniformly on on D, then f is continuous on D.

Theorem

Suppose that (f_n) is a sequence of continuous functions on a set D containing a contour Γ. If $f_n \to f$ uniformly on D, then f is continuous on D—and hence on Γ—and

$$\lim_n \int_\Gamma f_n(z)\, dz = \int_\Gamma f(z)\, dz.$$

The Safety of our Complex World

Theorem

Suppose that (f_n) is a sequence of analytic functions on a domain D. If $f_n \to f$ uniformly on D, then f is analytic on D.

Theorem (Taylor Series are Unique)

Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a power series with radius of convergence $R > 0$. Then

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n \tag{$\dagger$}$$

is analytic in $D = \{\, z : |z-z_0| < R \,\}$. Moreover,

$$a_n = \frac{f^{(n)}(z_0)}{n!}.$$

Therefore, ($\dagger$) is the Taylor series for f about z_0.

An Example

Example

Find the MacLaurin series for $f(z) = z\cos(z^2)$.

Example

Let $f(z) = z\cos(z^2)$. What is $f^{(2021)}(0)$?

Example

$$g(z) = \begin{cases} \frac{z-\sin(z)}{z^3} & \text{if } z \neq 0\text{, and} \\ \frac{1}{6} & \text{if } z = 0 \end{cases}$$

is an entire function.