Math 43: Spring 2020
Lecture 19 Part I

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Definition

A series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

with $a_n, z_0 \in \mathbf{C}$ is called a power series centered at z_0.

Example

Every Taylor series about z_0 is a power series centered at z_0.

General Nonsense

Theorem

Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ *be a power series centered at* z_0*. Then there is a* $0 \le R \le \infty$ *such that*

1. *the series converges absolutely if* $|z-z_0| < R$*,*
2. *The series diverges if* $|z-z_0| > R$*, and*
3. *the series converges uniformly on any closed subdisk* $D_r = \{\, z : |z-z_0| \le r \,\}$ *provided* $0 < r < R$*.*

Proof.

This is a homework problem using Lemma 2 in the text and some hints. □

Remark

Naturally, we call R the radius of convergence of $\sum_{n=0}^{\infty} a_n(z-z_0)^n$.

A Picture to Remember

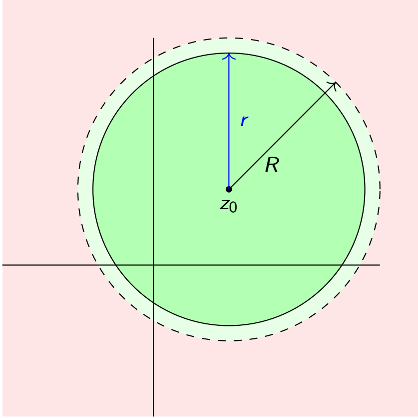


Figure: The Radius of Convergence of a power series $\sum_{n=0}^{\infty} a_n(z - z_0)$

Uniform is Good

Theorem

Suppose that (f_n) is a sequence of continuous complex-valued functions on a set $D \subset \mathbf{C}$. If $f_n \to f$ uniformly on on D, then f is continuous on D.

Proof.

Fix $z_0 \in D$. Then given $\epsilon > 0$ we need to find $\delta > 0$ so that $|z - z_0| < \delta$ implies $|f(z) - f(z_0)| < \epsilon$. But we can find N such that N implies that $|f_N(z) - f(z)| < \frac{\epsilon}{3}$ for all $z \in D$. Since f_N is assumed to be continuous at z_0, there is a $\delta > 0$ such that $|z - z_0| < \delta$ implies $|f_N(z) - f_N(z_0)| < \frac{\epsilon}{3}$. Then if $|z - z_0| < \delta$, we have

$$\begin{aligned} |f(z) - f(z_0)| &\leq |f(z) - f_N(z)| + |f_N(z) - f_N(z_0)| \\ &\qquad + |f_N(z_0) - f(z_0)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

□

Even Better

Theorem

Suppose that (f_n) is a sequence of continuous functions on a set D containing a contour Γ. If $f_n \to f$ uniformly on D, then f is continuous on D—and hence on Γ—and

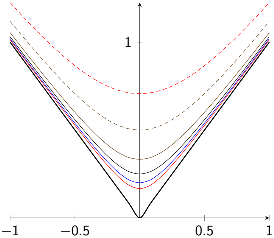
$$\lim_n \int_\Gamma f_n(z)\,dz = \int_\Gamma f(z)\,dz.$$

Proof.

The limit f is continuous by the previous result. Let $\epsilon > 0$. Let N be such that $n \geq N$ implies that $|f_n(z) - f(z)| < \frac{\epsilon}{\ell(\Gamma)+1}$ for all $z \in D$. Then

$$\begin{aligned} \left| \int_\Gamma f_n(z)\,dz - \int_\Gamma f(z)\,dz \right| &\leq \int_\Gamma |f_n(z) - f(z)|\,dz \\ &\leq \frac{\epsilon}{\ell(\Gamma)+1}\ell(\Gamma) < \epsilon. \end{aligned}$$

□



Let $f_n : [-1,1] \to \mathbf{R}$ be given by $f_n(x) = \sqrt{\frac{1}{n^2} + x^2}$. Then f is differentiable—smooth in fact. Moreover,
$x^2 \le f_n(x)^2 = \frac{1}{n^2} + x^2 \le \big(|x| + \frac{1}{n}\big)^2$
implies that $|x| \le f_n(x) \le |x| + \frac{1}{n}$.

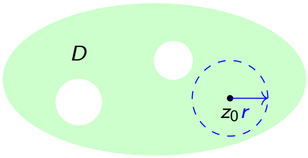
Hence if $f(x) = |x|$, then $f_n \to f$ uniformly on $[-1,1]$. Sadly, $f(x) = |x|$ is not differentiable at $x = 0$!

Remark (Advanced)

Remarkably, Weierstrass proved in the 19th century that the uniform limit of smooth functions, while continuous, need not be differentiable at a single point! Of course, we can't give a formula for such a function, but they exist in abundance!

Theorem

Suppose that (f_n) is a sequence of analytic functions on a domain D. If $f_n \to f$ uniformly on D, then f is analytic on D.



Fix $z_0 \in D$. Since z_0 is arbitrary, it will suffice to see that $f'(z_0)$ exists. Since D is open, there is a $r > 0$ such that $B_r(z_0) \subset D$. Thus we can replace D by $B_r(z_0)$ and assume that the D is simply connected. Let Γ be any closed contour in $D = B_r(z_0)$.

Since $f_n \to f$ uniformly, f is continuous and

$$\int_\Gamma f(z)\,dz = \lim_n \int_\Gamma f_n(z)\,dz. \qquad (\ddagger)$$

Return

Proof Continued

Proof.

Since D is simply connected and each f_n is analytic on D, each of the integrals on the right of (‡) are zero by the Cauchy Integral Theorem. Hence the integral of f is also zero. Since f is continuous and Γ is any closed contour in D, f is analytic in D by Morera's Theorem. Hence $f'(z_0)$ exists. □

More Good Stuff to Come

Remark

We will apply these impressive results to power series in the second part of the lecture.

Time for a Break.