Math 43: Spring 2020
Lecture 19 Part II

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Theorem (Taylor Series are Unique)

Let $\sum_{n=0}^{\infty} a_n(z - z_))^n$ *be a power series with radius of convergence* $R > 0$. *Then*

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n \tag{$\dagger$}$$

is analytic in $D = \{\, z : |z - z_0| < R \,\}$. *Moreover,*

$$a_n = \frac{f^{(n)}(z_0)}{n!}.$$

Therefore, (†) is the Taylor series for f *about* z_0.

Proof

Proof.

To see that f is analytic in all of D, it will suffice to see that f is analytic in $D_r = \{ z : |z - z_0| < r \}$ for any $0 < r < R$. Let $F_n(z) = \sum_{k=0}^{n} a_k(z - z_0)^k$ be the n^{th} partial sum of (†). We know from our general theorem on power series that $F_n \to f$ uniformly on D_r. Since each F_n is a polynomial, it is entire and hence analytic on D_r. By what we proved earlier, this means f is analytic on D_r. This proves the first assertion.

Let C_r be the positively oriented circle $|z - z_0| = r$. Fix $m \in \mathbf{N}$. Then if $w \in C_r$,

$$\left| \frac{f(w)}{(w - z_0)^{m+1}} - \frac{F_n(w)}{(w - z_0)^{m+1}} \right| = \frac{1}{r^{m+1}} \big| f(w) - F_n(w) \big|.$$

It follows that $\dfrac{F_n(w)}{(w - z_0)^{m+1}} \to \dfrac{f(w)}{(w - z_0)^{m+1}}$ uniformly on C_r.

Proof Continued

Proof.

Therefore

$$\begin{aligned}\int_{C_r} \frac{f(w)}{(w-z_0)^{m+1}}\,dw &= \lim_{n\to\infty} \int_{C_r} \frac{F_n(w)}{(w-z_0)^{m+1}}\,dw \\ &= \lim_{n\to\infty} \sum_{k=0}^{n} a_k \int_{C_r} \frac{(w-z_0)^k}{(w-z_0)^{m+1}}\,dw \\ &= 2\pi i \cdot a_m.\end{aligned}$$

But the left-hand side is

$$\frac{2\pi i}{n!} f^{(m)}(z_0).$$

Hence

$$a_m = \frac{f^{(m)}(z_0)}{m!}. \qquad \square$$

An Example

Example

Find the MacLaurin series for $f(z) = z\cos(z^2)$.

Solution.

Repeated differentiation would not end well. But for all w,

$$\cos(w) = 1 - \frac{w^2}{2} + \frac{w^4}{4!} - \cdots = \sum_{n=0}^{\infty}(-1)^n\frac{w^{2n}}{(2n)!}.$$

Hence for any $z \in \mathbf{C}$ we have

$$z\cos(z^2) = z\sum_{n=0}^{\infty}(-1)^n\frac{(z^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n\frac{z^{4n+1}}{(2n)!} \quad (1)$$
$$= z - \frac{z^5}{2} + \frac{z^9}{4!} - \cdots .$$

By the previous result, this must be the MacLaurin series for $f(z) = z\cos(z^2)$. □

Example Continued

Example

Let $f(z) = z\cos(z^2)$. What is $f^{(2021)}(0)$?

Solution.

Let $f(z) = a_0 + a_1 z + \cdots$ as on the previous slide. Then

$$a_{2021} = \frac{f^{(2021)}(0)}{(2021)!}.$$

But $2021 = 4(505) + 1$. Hence, using (1)

$$a_{2021} = (-1)^{505} \cdot \frac{1}{(2 \cdot 505)!}.$$

Therefore

$$f^{(2021)}(0) = -\frac{(2021)!}{(1010)!}. \quad \square$$

Another Example

Let $f(z) = z - \sin(z)$. Then for all z,

$$\begin{aligned} f(z) &= z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right) \\ &= \frac{z^3}{3!} - \frac{z^5}{5!} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^{2n+1}}{(2n+1)!} \end{aligned}$$

Hence if $z \neq 0$,

$$\frac{z - \sin(z)}{z^3} = \frac{1}{6} - \frac{z^2}{5!} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^{2n-2}}{(2n+1)!}.$$

But the right-hand side is an entire function. Hence

$$g(z) = \begin{cases} \frac{z - \sin(z)}{z^3} & \text{if } z \neq 0, \text{ and} \\ \frac{1}{6} & \text{if } z = 0 \end{cases}$$

is an entire function.

Enough for Today

Remark

This completes §5.3 and the material for the midterm on Wednesday. Get a head start and have some good questions ready for class on Monday and Wednesday as well as office hours on Tuesday.

That is all for today.