Math 43: Spring 2020
Lecture 2 Part 1

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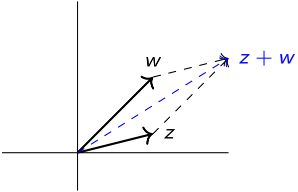
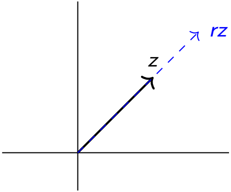
April 1, 2020

The Geometry of Complex Addition

Back in grade school, we were taught that addition involved "trips on the number line". For example, the sum $4 - 5 := 4 + (-5)$ was a trip of 4 units to the right followed by 5 units to the left. Thank goodness we're past that now. But things are much more interesting in 2-dimensions. Complex addition is just vector addition.



(a) Vector Addition



(b) Multiplication by $r > 1$

Multiplication by a real constant r is just scalar multiplication of vectors.

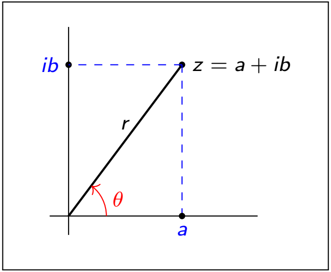
Polar Coordinates

To understand what complex multiplication "looks like", we need to recall what polar coordinates are.

If $z = a + ib$, then its polar coordinates (r, θ) are determined as follows. We let $r = |z| = \sqrt{a^2 + b^2}$. The angle θ is determined by the equations

$$\cos(\theta) = \frac{a}{r} \quad \text{and} \quad \sin(\theta) = \frac{b}{r}.$$

Many texts prefer using $\tan(\theta) = \frac{b}{a}$ with appropriate noises about what to do if $a = 0$. Note that $\theta = \arctan\left(\frac{b}{a}\right)$ only if $a > 0$.



Remark

Keep in mind that θ is only determined up to an integer multiple of 2π.

Polar Form of a Complex Number

Note that if $z = a + ib$ has polar coordinates (r, θ), then $a = r\cos(\theta)$ and $b = r\sin(\theta)$. Hence $z = rcos(\theta) + ir\sin(\theta) = r\big(\cos(\theta) + i\sin(\theta)\big)$. For reasons best known to the authors of our text, they define $\operatorname{cis}(\theta) = \cos(\theta) + i\sin(\theta)$. Then we can write $z = r\operatorname{cis}(\theta)$.

Definition

Let z be a nonzero complex number with polar coordinates (r, θ). Then we call $r\operatorname{cis}(\theta)$ the polar form of z. Furthermore, we call θ an argument of z. The set of arguments of z is denoted by $\arg(z)$.

Remark

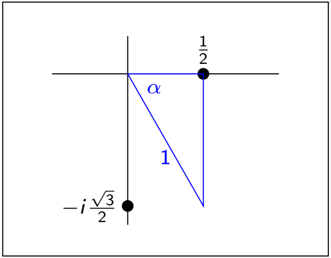
I've empolyed the "cis" notation for consistency with the textbook. We shall dispose of it as soon as we can and replace it with something better.

An Example of Polar Form

Let $z = 1 - i\sqrt{3}$. We immediately see that $r = \sqrt{1+3} = 2$. To figure out θ, we consider

$$\cos(\theta) = \frac{1}{2} \quad \text{and} \quad \sin(\theta) = -\frac{\sqrt{3}}{2}.$$

To figure
out what θ is, we use a "reference triangle" and a little right-triangle trigonometry. Then $\alpha = \frac{\pi}{3}$.
Hence $\theta = -\frac{\pi}{3} + 2\pi k$, with $k \in \mathbf{Z}$.
Then $\arg(z) = \{\, -\frac{\pi}{3} + 2\pi k : k \in \mathbf{Z}\,\}$,
and the following
are polar forms of $z = 1 - i\sqrt{3}$:



$$2\,\mathrm{cis}\Big(-\frac{\pi}{3}\Big) \quad \text{and} \quad 2\,\mathrm{cis}\Big(\frac{5\pi}{3}\Big).$$

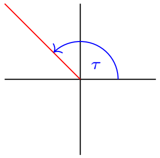
Of course, there are infinitely many polar forms of z.

The Argument

Remark

The map $z \mapsto \arg(z)$ is not a function in the usual sense. This is because $\arg(z)$ is a set not a simply a number. I'll refer to $z \mapsto \arg(z)$ as a set-valued function. The text calls it a "multivalued-function" which in my opinion is a oxymoron.

To get a bona fide real-valued function we have to make a choice for each non-zero z. As a crude example, if $\tau \in \mathbf{R}$, then the text defines $\arg_\tau(z)$ to be the unique element in the intersection $\arg(z) \cap (\tau, \tau + 2\pi]$. Note that $\arg_\tau$ is not defined if $z = 0$, and has a jump discontinuity along the ray $\{ z \in \mathbf{C} : \tau \in \arg(z) \}$.

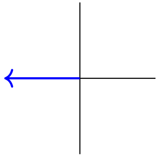


The Principal Arugument Function

Definition

If $z \neq 0$, then we define the principal value of the argument of z to be $\operatorname{Arg}(z) := \arg_{-\pi}(z)$

Note that $\operatorname{Arg}(i) = \frac{\pi}{2}$ and $\operatorname{Arg}(-1-i) = \frac{-3\pi}{4}$. Also, $\operatorname{Arg}(1 - i\sqrt{3}) = -\frac{\pi}{3}$. But $\arg_0(1 - i\sqrt{3}) = \frac{5\pi}{3}$.



TAKE NOTE: There is nothing special about the principal value, $\operatorname{Arg}(z)$, of z. It is just a choice that may, or may not, be convenient in the moment. In fact all the functions $z \mapsto \arg_\tau(z)$ are to be regarded with suspicion and dragged out only under duress.

Back to High School Trigonometry

If you had a good high school trigonometry course, then you saw the sum formulas for sine and cosine:

$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B) \quad \text{and}$$
$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B).$$

In all honesty, it is not so easy to give easy proofs of these identities unless A, B, and $A+B$ are acute angles, but we will accept them as given.

The Key Result

Theorem

Let $z = r\,\mathrm{cis}(\theta)$ and $w = \rho\,\mathrm{cis}(\varphi)$. Then we have the following formulas.

$$zw = r\rho\,\mathrm{cis}(\theta+\varphi) \quad \textit{and} \quad \frac{z}{w} = \frac{r}{\rho}\,\mathrm{cis}(\theta-\varphi).$$

Proof.

We calculate

$$\begin{aligned} zw &= r\rho\big(\cos(\theta) + i\sin(\theta)\big)\big(\cos(\varphi)+i\sin(\varphi)\big) \\ &= r\rho\big[\cos(\theta)\cos(\varphi) - \sin(\theta)\sin(\varphi) \\ &\qquad\qquad + i\big(\cos(\theta)\sin(\varphi)+\sin(\theta)\cos(\varphi)\big)\big] \\ &= r\rho\big[\cos(\theta+\varphi) + i\sin(\theta+\varphi)\big] \\ &= r\rho\,\mathrm{cis}(\theta+\varphi). \end{aligned}$$

This establishes the first equation.

The Proof Continued

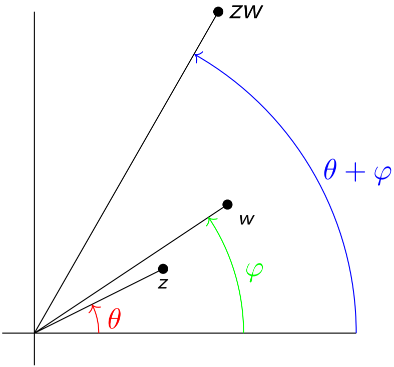
Proof Continued.

For the second equation, consider the special case

$$\begin{aligned}
\frac{1}{w} &= \frac{1}{\rho\big(\cos(\varphi) + i\sin(\varphi)\big)} \\
&= \frac{1}{\rho} \cdot \frac{1}{\cos(\varphi) + i\sin(\varphi)} \cdot \frac{\cos(\varphi) - i\sin(\varphi)}{\cos(\varphi) - i\sin(\varphi)} \\
&= \frac{1}{\rho}\big(\cos(\varphi) - i\sin(\varphi)\big) = \frac{1}{\rho}\big(\cos(-\varphi) + i\sin(-\varphi)\big) \\
&= \frac{1}{\rho}\operatorname{cis}(-\varphi).
\end{aligned}$$

Now $\frac{z}{w} = z\frac{1}{w} = (r\operatorname{cis}(\theta))(\frac{1}{\rho}\operatorname{cis}(-\varphi))$ and we can use the first equation to establish the second. $\square$

Let $z = r\,\mathrm{cis}(\theta)$ and let $w = \rho\,\mathrm{cis}(\varphi)$. Then the previous Theorem implies that $zw = r\rho\,\mathrm{cis}(\theta + \varphi)$. Thus multiplication of w by z means that zw is obtained by rotating w by θ and stretching its length by a factor of r. Now suppose that $z = \mathrm{cis}(\theta)$. Geometrically, we happens to integral powers of z—that is, z^n? Well, since $|z| = 1$, $z^n = \mathrm{cis}(n\theta)$ is always on the unit circle and gets rotated by θ radians counterclockwise with each power of z. Notice that if n is negative, then we rotate clockwise!



We get some important corollaries from our theorem on complex multiplication: $r\operatorname{cis}(\theta) \cdot \rho\operatorname{cis}(\varphi) = r\rho\operatorname{cis}(\theta + \varphi)$.

Corollary

If z and w are complex numbers, then $|zw| = |z||w|$.

Corollary

If z and w are nonzero complex numbers, then

$$\begin{aligned}\arg(zw) &= \arg(z) + \arg(w)\\ &= \{\, \theta + \varphi : \theta \in \arg(z) \textit{ and } \varphi \in \arg(w) \,\}.\end{aligned}$$

The Argument is Complicated

Remark

The pretty formula $\arg(zw) = \arg(z) + \arg(w)$ is all well and good, but it doesn't usually work when we force functions like Arg into play.

Note that $\mathrm{Arg}(i) = \frac{\pi}{2}$, $\mathrm{Arg}(-1) = \pi$, and $\mathrm{Arg}(-i) = -\frac{\pi}{2}$.

But

$$-\frac{\pi}{2} = \mathrm{Arg}(-i) = \mathrm{Arg}((-1)i) \neq \mathrm{Arg}(-1) + \mathrm{Arg}(i) = \frac{3\pi}{2}.$$

Time for a Break!