Math 43: Spring 2020
Lecture 2 Part 2

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April 1, 2020

Exponential Functions

- Single variable calculus gets much more interesting once we have some functions around that aren't simply polynomials or rational functions.
- One of the most interesting and fundamental is the natural exponential function $f(x) = e^x$.
- But we can't "just change the x to a z". What would e^i or even 2^i mean?
- For motivation, consider what sort of properties we want our complex exponential function $z \mapsto e^z$ to have once we make sense of the symbol e^z.
- At the very least, we want $e^{z+w} = e^z e^w$ when z and w are arbitrary complex numbers and not just positive integers. After all, this is the way exponents are supposed to work.
- Given that, we want $e^{x+iy} = e^x e^{iy}$.

A Little Imagination

The previous slide allows us to guess a good definition for e^z if we can guess what e^{iy} should be when $y \in \mathbf{R}$.
We should all be familiar with the MacLaurin series for the natural exponential function:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

So proceeding formally, without justification, we guess that we should have

$$\begin{aligned} e^{iy} &= 1 + iy - \frac{y^2}{2!} - i\frac{y^3}{3!} + \frac{y^4}{4!} + \cdots \\ &= \Bigl(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots\Bigr) + i\Bigl(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots\Bigr) \\ &= \cos(y) + i\sin(y) \\ &= \operatorname{cis}(y) \end{aligned}$$

The Complex Exponential Function

With only the previous speculation as motivation, we define the complex exponential function as follows.

Definition

If $z = z + iy \in \mathbf{C}$, then we define

$$e^z = e^x\big(\cos(y) + i\sin(y)\big) = e^x \operatorname{cis}(y).$$

Remark (Death to cis)

Now that we have what will prove to be a good definition of e^z in hand, we will always write e^{iy} in place of $\operatorname{cis}(y)$. It should be tossed aside as low brow childish tripe such as thinking of addition as walks on the number line.

This means the polar form of z with polar coordinates (r, θ) is now $z = re^{i\theta}$. Then, for example, $re^{i\theta}\rho e^{i\varphi} = r\rho e^{i(\theta+\varphi)}$.

But is it a Good Definition??

Having pulled this definition out of thin air, we need to prove that our exponential function has some of the properties we want.

Theorem

If $z, w \in \mathbf{C}$, then

$$\text{(a)} \quad e^z e^w = e^{z+w} \qquad \textit{and} \qquad \text{(b)} \quad \frac{e^z}{e^w} = e^{z-w}.$$

Corollary

For all $z \in \mathbf{C}$, we have

$$\text{(a)} \quad e^{-z} = \frac{1}{e^z} \qquad \textit{and} \qquad \text{(b)} \quad \left(e^z\right)^n = e^{nz} \quad \textit{for all } n \in \mathbf{Z}.$$

DeMoivre's Formula

Least you think we haven't done anything, consider the following.

Corollary

If $\theta \in \mathbf{R}$ and $n \in \mathbf{Z}$, then

$$(e^{i\theta})^n = e^{in\theta}.$$

Ok, still not impressed?

Remark

This is a lot cooler looking if you write it out as

$$\underbrace{\big(\cos(\theta) + i\sin(\theta)\big)^n = \cos(n\theta) + i\sin(n\theta)}_{\text{DeMovire's Formula}}$$

Powers

Example

Simplify $(1-i)^{14}$.

Solution.

$$\begin{aligned}(1-i)^{14} &= (\sqrt{2}e^{-\frac{\pi}{4}})^{14} = 2^7 e^{-i\frac{7\pi}{2}} \\ &= 128e^{i(\frac{\pi}{2}-4\pi)} = 128e^{i\frac{\pi}{2}} \\ &= \boxed{128i}.\end{aligned}$$

$\square$

Trigonometric Identities

Example

Suppose that $\cos(\theta) = \frac{1}{5}$. What is $\cos(3\theta)$?

Solution

We can't work out what θ is. Instead, We try to write $\cos(3\theta)$ *in terms of* $\cos(\theta)$. *Using DeMoivre's Formula,*

$$\begin{aligned}\cos(3\theta) &= \mathrm{Re}\big(\cos\theta + i\sin\theta\big)^3 \\ &= \cos^3(\theta) - 3\cos(\theta)\sin^2(\theta) \\ &= \cos^3(\theta) - 3\cos(\theta)(1-\cos^2(\theta)) \\ &= 4\cos^3(\theta) - 3\cos(\theta).\end{aligned}$$

Thus $\cos(3\theta) = 4\left(\frac{1}{5}\right)^3 - 3\left(\frac{1}{5}\right) = \boxed{-\frac{71}{125}}$.