Math 43: Spring 2020
Lecture 20 Summary

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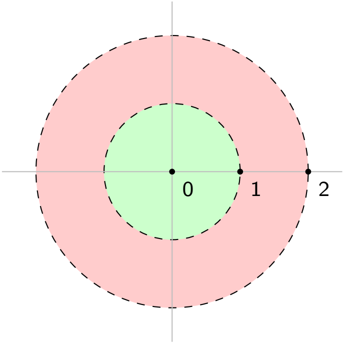
This Week

1. We should be recording.
2. Who's turn is it for extra credit?
3. I plan to email the midterm out right after this class meeting is over. Last time I got "distracted by foolishness" and had to be reminded.
4. You solutions should be uploaded by class meeting time on Friday.



Consider $f(z) = \frac{1}{(z-1)(z-2)}$. Note that this function is analytic in the annuli $A_1 = \{ z : 0 < |z| < 1 \}$, $A_2 = \{ z : 1 < |z| < 2 \}$, and $A_3 = \{ z : 2 < |z| \}$. (There is no harm in omitting 0 from A_1.) Then we used geometric series to verify the following.

1. In A_1, $f(z) = \sum_{n=0}^{\infty} \Bigl(1 - \frac{1}{2^{n+1}}\Bigr) z^n$.
2. In A_2, $f(z) = \sum_{n=0}^{\infty} \frac{-1}{2^{n+1}} z^n + \sum_{n=1}^{\infty} \frac{-1}{z^n}$.
3. And it is an exercise to verify that in A_3, $f(z) = \sum_{j=2}^{\infty} \frac{2^{j-1}-1}{z^j}$.

Laurent Series

Definition

A series of the form $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ is called a Laurent series about z_0. We will almost always write this in the form

$$\underbrace{\sum_{n=0}^{\infty} a_n(z-z_0)^n}_{\text{power series bit}} + \underbrace{\sum_{j=1}^{\infty} \frac{b_j}{(z-z_0)^j}}_{\text{singular bit}}$$

where $b_j = a_{-j}$.

Theorem

Suppose that

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{j=1}^{\infty} \frac{b_j}{(z - z_0)^j} \qquad (\dagger)$$

is a Laurent series about z_0*. Then either* $(\ddagger)$ *converges nowhere or there are* $0 \leq r \leq R \leq \infty$ *such that the series converges absolutely if*

$$z \in A := \{\, z : r < |z - z_0| < R \,\},$$

and such that the convergence is uniform in every sub-annulus

$$A' = \{\, z : r' \leq |z - z_0| \leq R' \,\}$$

provided that $r < r' < R' < R$.

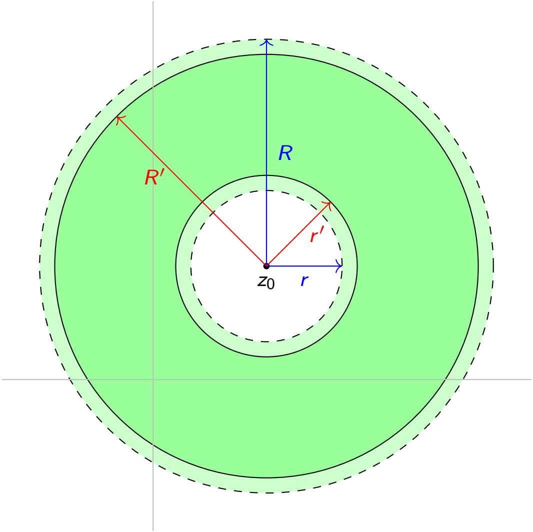


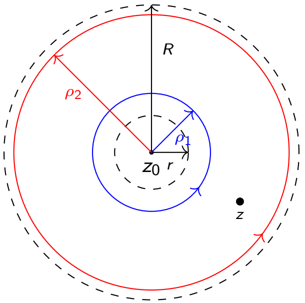
Figure: Convergence for a Laurent Series

Cauchy Again

Theorem (Cauchy's Integral Formula for a Annulus)

Suppose that f is analytic in $A = \{\, z : 0 \le r < |z - z_0| < R \le \infty \,\}$. We let C_ρ denote the positively oriented circle $|z - z_0| = \rho$. Then if $r < \rho_1 < \rho_2 < R$ and if $\rho_1 < |z - z_0| < \rho_2$, we have

$$f(z) = \frac{1}{2\pi i} \int_{C_{\rho_2}} \frac{f(w)}{w - z}\, dw - \frac{1}{2\pi i} \int_{C_{\rho_1}} \frac{f(w)}{w - z}\, dw$$



Theorem (Laurent's Theorem)

Suppose that f is analytic in an annulus $A = \{ z : 0 \le r < |z - z_0| < R \le \infty \}$ with $r < R$. Then there are complex numbers a_n and b_j such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{j=1}^{\infty} \frac{b_j}{(z - z_0)^j} \quad \text{for all } z \in A. \qquad (\ddagger)$$

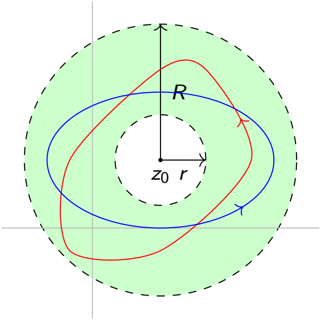
Moreover, if C is any positively oriented simple closed contour in A with z_0 in its interior, then

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w - z_0)^{n+1}} \, dw \quad \text{and}$$

$$b_j = \frac{1}{2\pi i} \int_C f(w)(w - z_0)^{j-1} \, dw.$$

We call ($\ddagger$) the Laurent series for f in A about z_0.

The Choice of the Countor C



The Deformation Invariance Theorem allows us to pick any positively oriented closed contour in A to compute the coefficients a_n and b_j in the Laurent series for f in A about z_0.

Remark

1. Note that in general, we can't expect
$$a_n = \frac{f^{(n)}(z_0)}{n!}$$
as the hypotheses of the Cauchy Integral Formula are not met. Even worse, f may not even be defined at z_0 let alone analytic there.
2. Since (‡) is a convergent Laurent series, its convergence is uniform in any subannulus
$$A' = \{\, z : r < r' \leq |z - z_0| \leq R' < R \,\}.$$
3. We will use this uniform convergence in the proof in the form of the following lemma.

Lemma

Suppose that f_n is continuous on set D and that $\sum_{n=0}^{\infty} f_n(z)$ converges uniformly to f on D and that Γ is a contour in D. Then f is continuous on D and

$$\int_{\Gamma} f(z)\,dz = \sum_{n=0}^{\infty} \int_{\Gamma} f_n(z)\,dz.$$

Theorem

Suppose that

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{j=1}^{\infty} \frac{b_j}{(z - z_0)^j} \tag{$\dagger$}$$

converges in $A = \{\, z : 0 \le r < |z - z_0| < R \le \infty \,\}$ with $r < R$. Then f is analytic in A and ($\dagger$) is the Laurent series for f in A about z_0.