Math 43: Spring 2020
Lecture 20 Part I

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Fun with Geometric Series

Consider $f(z) = \dfrac{1}{(z-1)(z-2)}$. This function is analytic in $D = \mathbf{C} \setminus \{1,2\}$. In particular, it is analytic in $B_1(0)$ and has a MacLaurin series with radius of convergence $R = 1$. We can find it easily using what we know about geometric series. Thus if $|z| < 1$, we have

$$\begin{aligned} f(z) &= \frac{A}{z-1} + \frac{B}{z-2} = -\frac{1}{z-1} + \frac{1}{z-2} \\ &= \frac{1}{1-z} - \frac{1}{2-z} = \frac{1}{1-z} - \frac{1}{2} \cdot \frac{1}{1-\frac{z}{2}} \\ &= \sum_{n=0}^{\infty} z^n - \frac{1}{2} \sum_{n=0}^{\infty} \Big(\frac{z}{2}\Big)^n \\ &= \sum_{n=0}^{\infty} \Big(1 - \frac{1}{2^{n+1}}\Big) z^n. \end{aligned}$$

Analytic in an Annulus

But f is also analytic in the annulus $A = \{ z : 1 < |z| < 2 \}$. Now we play the same game, but with A in mind.

$$f(z) = -\frac{1}{z-1} + \frac{1}{z-2} = -\frac{1}{2-z} - \frac{1}{z-1}$$
$$= -\frac{1}{2}\Big(\frac{1}{1-\frac{z}{2}}\Big) - \frac{1}{z}\Big(\frac{1}{1-\frac{1}{z}}\Big)$$

which, since $1 < |z| < 2$, is

$$= -\frac{1}{2}\sum_{n=0}^{\infty}\Big(\frac{z}{2}\Big)^n - \frac{1}{z}\sum_{n=0}^{\infty}\Big(\frac{1}{z}\Big)^n$$
$$\sum_{n=0}^{\infty}\frac{-1}{2^{n+1}}z^n + \sum_{n=1}^{\infty}\frac{-1}{z^n}.$$

Note that in the annulus A, we have written $f(z)$ as the sum of a series of the form $\displaystyle\sum_{n=-\infty}^{\infty} a_n z^n$. Return

Laurent Series

Definition

A series of the form $\displaystyle\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ is called a Laurent series about z_0. We will almost always write this in the form

$$\underbrace{\sum_{n=0}^{\infty} a_n(z-z_0)^n}_{\text{power series bit}} + \underbrace{\sum_{j=1}^{\infty} \frac{b_j}{(z-z_0)^j}}_{\text{singular bit}}$$

where $b_j = a_{-j}$. ▸ Compare ▸ Return

Let R be the radius of convergence of what I cavalierly called the "power series bit" on the ▸ previous slide. Hence the first sum converges if $|z - z_0| < R$. But if we let $\xi = \frac{1}{z-z_0}$, then the "singular bit" is

$$\sum_{j=1}^{\infty} \frac{b_j}{(z - z_0)^j} = \sum_{j=1}^{\infty} b_j \xi^j.$$

If R' is the radius of convergence of the second sum, the the singular bit converges if $\left|\frac{1}{z-z_0}\right| < R'$. Equivalently, the singular bit converges if $|z - z_0| > \frac{1}{R'}$.

Using this sort of analysis, we can prove the following result about Laurent series.

Theorem

Suppose that

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{j=1}^{\infty} \frac{b_j}{(z-z_0)^j} \qquad (\dagger)$$

is a Laurent series about z_0. Then either $(\dagger)$ converges nowhere or there are $0 \le r \le R \le \infty$ such that the series converges absolutely if

$$z \in A := \{\, z : r < |z-z_0| < R \,\},$$

and such that the convergence is uniform in every sub-annulus

$$A' = \{\, z : r' \le |z-z_0| \le R' \,\}$$

provided that $r < r' < R' < R$.

Picture

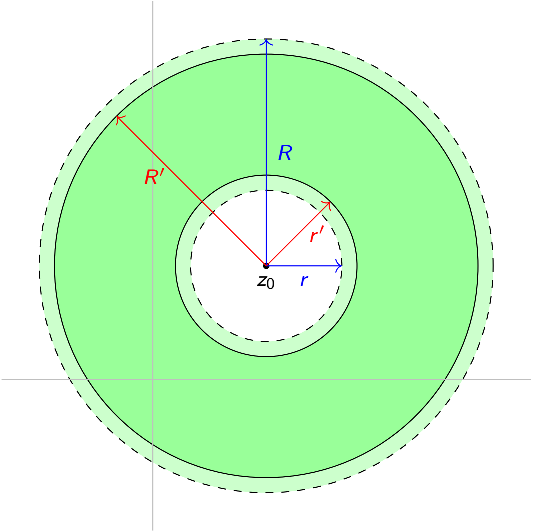


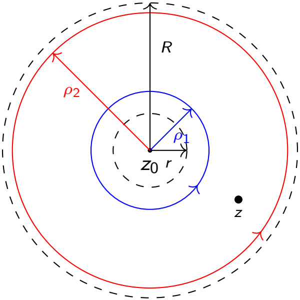
Figure: Convergence for a Laurent Series

Cauchy Again

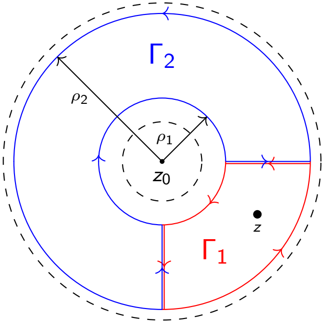
Theorem (Cauchy's Integral Formula for a Annulus)

Suppose that f is analytic in $A = \{\, z : 0 \le r < |z - z_0| < R \le \infty \,\}$. We let C_ρ denote the positively oriented circle $|z - z_0| = \rho$. Then if $r < \rho_1 < \rho_2 < R$ and if $\rho_1 < |z - z_0| < \rho_2$, we have

$$f(z) = \frac{1}{2\pi i} \int_{C_{\rho_2}} \frac{f(w)}{w - z}\, dw - \frac{1}{2\pi i} \int_{C_{\rho_1}} \frac{f(w)}{w - z}\, dw$$



Proof



Suppose that $\rho_1 < |z - z_0| < \rho_2$. Then we construct the two contours Γ_1 and Γ_2 as drawn at right.

The point is that if g is any continuous function on union of the two circles C_{ρ_2} and C_{ρ_1}, then because the contributions on the overlapping line segments cancel,

$$\int_{\Gamma_1} g(w)\,dw + \int_{\Gamma_2} g(w)\,dw = \int_{C_{\rho_2}} g(w)\,dw - \int_{C_{\rho_1}} g(w)\,dw. \quad (*)$$

▸ Return

Proof Continued

Proof.

Since f is analtyic on and inside Γ_1 and z is inside of Γ_1, we have

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma_1} \frac{f(w)}{w - z} \, dw$$

by the Cauchy Integral Formula. On the other hand, $w \mapsto \frac{f(w)}{w-z}$ is analytic on and inside of Γ_2. Thus by the Cauchy Integral Theorem

$$\frac{1}{2\pi i} \int_{\Gamma_2} \frac{f(w)}{w - z} \, dw = 0.$$

Finish

Proof.

Hence

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma_1} \frac{f(w)}{w - z}\, dw + \frac{1}{2\pi i} \int_{\Gamma_2} \frac{f(w)}{w - z}\, dw$$

which, in view of $(*)$ ▸ $(*)$, is

$$= \frac{1}{2\pi i} \int_{C_{\rho_2}} \frac{f(w)}{w - z}\, dw - \frac{1}{2\pi i} \int_{C_{\rho_1}} \frac{f(w)}{w - z}\, dw$$

as required. □

Enough for Now

Remark

After the break, we will make good use of the Cauchy Integral formula for an Annulus.

Time for a Break.