Math 43: Spring 2020
Lecture 20 Part II

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Theorem (Laurent's Theorem)

Suppose that f is analytic in an annulus $A = \{\, z : 0 \leq r < |z - z_0| < R \leq \infty \,\}$ with $r < R$. Then there are complex numbers a_n and b_j such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{j=1}^{\infty} \frac{b_j}{(z - z_0)^j} \quad \text{for all } z \in A. \qquad (\ddagger)$$

Moreover, if C is any positively oriented simple closed contour in A with z_0 in its interior, then

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w - z_0)^{n+1}}\, dw \quad \text{and}$$

$$b_j = \frac{1}{2\pi i} \int_C f(w)(w - z_0)^{j-1}\, dw.$$

Remark

1. We call (‡) the Laurent series for f in A about z_0.
2. The formulas for a_n and b_j are independent of our choice of C by the Deformation Invariance Theorem.
3. Note that in general, we can't expect

$$a_n = \frac{f^{(n)}(z_0)}{n!}$$

as the hypotheses of the Cauchy Integral Formula are not met. Even worse, f may not even be defined at z_0 let alone analytic there.

4. Since (‡) is a convergent Laurent series, its convergence is uniform in any subannulus

$$A' = \{\, z : r < r' \leq |z - z_0| \leq R' < R \,\}.$$

5. We will use this uniform convergence in the proof in the form of the following lemma.

Lemma

Suppose that f_n is continuous on set D and that $\sum_{n=0}^{\infty} f_n(z)$ converges uniformly to f on D and that Γ is a contour in D. Then f is continuous on D and

$$\int_\Gamma f(z)\,dz = \sum_{n=0}^{\infty} \int_\Gamma f_n(z)\,dz.$$

Proof.

We already know that f must be continuous. By assumption the partial sums $F_n(z) = \sum_{k=0}^{n} f_k(z)$ converge uniformly to $f(z)$ for all $z \in D$. Therefore

$$\int_\Gamma f(z)\,dz = \lim_n \int_\Gamma F_n(z)\,dz = \lim_n \sum_{k=0}^{n} \int_\Gamma f_k(z)\,dz$$
$$= \sum_{k=0}^{\infty} \int_\Gamma f_k(z)\,dz. \qquad \square$$

Sketch of the Proof

Since the proof is very similar to that of Taylor's Theorem, I am only going to sketch the details. If $z \in A$, then there are $r < \rho_1 < \rho_2 < R$ such that $\rho_1 < |z - z_0| < \rho_2$. Then by Cauchy's Integral Formula for an annulus,

$$f(z) = \frac{1}{2\pi i} \int_{C_{\rho_2}} \frac{f(w)}{w - z}\, dw - \frac{1}{2\pi i} \int_{C_{\rho_1}} \frac{f(w)}{w - z}\, dw.$$

Now just as in the proof of Taylor's Theorem, if $w \in C_{\rho_2}$, then

$$\frac{f(w)}{w - z} = f(w) \frac{1}{w - z_0 - (z - z_0)} = \frac{f(w)}{w - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{w - z_0}}$$

which, since $\left|\frac{z-z_0}{w-z_0}\right| < 1$ if $w \in C_{\rho_2}$, is

$$= f(w) \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(w - z_0)^{n+1}} = \sum_{n=0}^{\infty} \frac{f(w)(z - z_0)^n}{(w - z_0)^{n+1}}.$$

Getting the Power Series Bit

Using this formula,

$$\frac{1}{2\pi i}\int_{C_{\rho_2}} \frac{f(w)}{w-z}\,dw = \frac{1}{2\pi i}\int_{C_{\rho_2}} \sum_{n=0}^{\infty} \frac{f(w)(z-z_0)^n}{(w-z_0)^{n+1}}\,dw$$

which, since our lemma applies, is

$$= \sum_{n=0}^{\infty} \int_{C_{\rho_2}} \frac{f(w)}{(w-z_0)^{n+1}}\,dw(z-z_0)^n$$

which, by our formula for the a_n, is

$$= \sum_{n=0}^{\infty} a_n(z-z_0)^n$$

On to C_{ρ_1}

Now if $w \in C_{\rho_1}$, we no longer have $\left|\frac{z-z_0}{w-z_0}\right| < 1$. In fact, now $\left|\frac{z-z_0}{w-z_0}\right| > 1$. So we work with the reciprocal. If $w \in C_{\rho_1}$, then

$$\frac{f(w)}{w-z} = -\frac{f(w)}{z-w} = -\frac{f(w)}{(z-z_0)-(w-z_0)} = -\frac{f(w)}{z-z_0}\frac{1}{1-\frac{w-z_0}{z-z_0}}$$

$$= -f(z)\sum_{n=0}^{\infty}\frac{(w-z_0)^n}{(z-z_0)^{n+1}}$$

$$= -f(w)\sum_{j=1}^{\infty}\frac{(w-z_0)^{j-1}}{(z-z_0)^j}.$$

Finish

Hence,

$$-\frac{1}{2\pi i}\int_{C_{\rho_1}}\frac{f(w)}{w-z}\,dw = \frac{1}{2\pi i}\int_{C_{\rho_1}}\sum_{j=1}^{\infty}\frac{f(w)(w-z_0)^{j-1}}{(z-z_0)^j}$$

which, by our lemma, is

$$=\sum_{j=1}^{\infty}\frac{1}{2\pi i}\int_{C_{\rho_1}} f(w)(w-z_0)^{j-1}\,dw\,\frac{1}{(z-z_0)^j}$$

which, given our formula for the b_j, is

$$=\sum_{j=1}^{\infty}\frac{b_j}{(z-z_0)^j}$$

This completes our sketch of the proof.

Uniqueness

Theorem

Suppose that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{j=1}^{\infty} \frac{b_j}{(z - z_0)^j} \tag{$\dagger$}$$

converges in $A = \{\, z : 0 \le r < |z - z_0| < R \le \infty \,\}$ *with* $r < R$. *Then* f *is analytic in* A *and* $(\dagger)$ *is the Laurent series for* f *in* A *about* z_0.

Proof.

The proof is similar to the one for power series. We will skip the details. □

Examples

We hardly ever use the formulas for the coefficients a_n and b_j for a Laurent series.

Example

Let $f(z) = \dfrac{\sin(z)}{z^5}$. Then f is analytic in the annulus $A = \{\, z : 0 < |z| \,\}$. Find the Laurent series.

Solution.

If $z \neq 0$, then

$$\begin{aligned} f(z) &= \frac{1}{z^5}\Big(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\Big) \\ &= \frac{1}{z^4} - \frac{1}{3! \cdot z^2} + \frac{1}{5!} - \frac{z^2}{7!} + \cdots \\ &= \frac{1}{z^4} - \frac{1}{6 \cdot z^2} + \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n+5)!}. \end{aligned}$$

By the previous theorem, this must be the Laurent series for f about 0. □

Another

Example

Let $g(z) = z^2 e^{\frac{1}{z}}$. Find the Laurent series about 0.

Solution.

$$\begin{aligned} g(z) &= z^2\Big(1 + \frac{1}{z} + \frac{1}{2!\cdot z^2} + \frac{1}{3!\cdot z^3} + \cdots\Big) \\ &= z^2 + z + \frac{1}{2} + \frac{1}{6z} + \cdots \\ &= z^2 + z + \frac{1}{2} + \sum_{j=1}^{\infty} \frac{1}{(j+2)!z^j}. \end{aligned}$$

$\square$

Enough

Remark

We will use Laurent series down the road to investigate the sorts of singularities that analytic functions can have.

That is enough for today.