Math 43: Spring 2020
Lecture 21 & 22 Summary

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Today

1. We should be recording!
2. You should have your graded exam back.
3. I emailed solutions out as well as information about your standing in the course.
4. You can find a hist-o-gram of the classes scores on the exam on the assignments page.
5. I will have office hours today 1:30 - 2:30 as well as my regular office hours on Tuesday if you want to talk about the course prior to the drop deadline on Wednesday.
6. I am also happy to talk about the exam anytime if you have questions or concerns.

Zeros

Definition

Suppose that f is analytic at z_0 and that $f(z_0) = 0$. Then we say that z_0 is a zero of order $m \geq 1$ for f if

$$0 = f(z_0) = f'(z_0) = \cdots = f^{(m-1)}(z_0) \quad \text{and} \quad f^{(m)}(z_0) \neq 0.$$

If $f^{(m)}(z_0) = 0$ for all $n \geq 0$, then we say that z_0 is a zero of infinite order.

Theorem

Suppose that f is analytic in a domain D. If f has a zero of infinite order in D, then f is identically zero in D.

Isolated Zeros

Theorem

Suppose that f is a non-constant analytic function in a domain D. If $z_0 \in D$ is a zero for f, then z_0 has finite order $m \geq 1$, and there is an analytic function g on D such that $g(z_0) \neq 0$ and such that

$$f(z) = (z - z_0)^m g(z) \quad \textit{for all } z \in D.$$

Corollary

If f is a non-constant analytic function on a domain D, then the zeros of f are isolated. That is, if $f(z_0) = 0$ for some $z_0 \in D$, then there is a $\delta > 0$ such that $f(z) \neq 0$ if $z \in B'_\delta(z_0) = \{ z : 0 < |z - z_0| < \delta \}$.

And Now Singularities

Definition

If f is analytic in $B'_R(z_0)$ for some $R > 0$, then we say that f has an isolated singularity at z_0

Remark

If f has an isolated singularity at z_0, then for some $R > 0$, f is analytic in the degenerate annulus $A = \{ z : 0 < |z - z_0| < R \}$. Hence f has a Laurent series in A about z_0:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{j=1}^{\infty} \frac{b_j}{(z - z_0)^j} \qquad (\ddagger)$$

for all $z \in A$. We know that the coefficients a_n and b_j appearing in $(\ddagger)$ depend only on f and z_0 and can be computed with respect to any positively oriented simple close contour in $B'_R(z_0)$ containing z_0 in its interior. Hence we call $(\ddagger)$ the Laurent series for f at z_0.

Classification of Singularities

Definition

Suppose that f has an isolated singularity at z_0 with Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{j=1}^{\infty} \frac{b_j}{(z - z_0)^j}$$

1. If $b_j = 0$ for all $j \geq 1$, then we call z_0 a removable singularity.
2. If $b_j = 0$ for all $j > m$ and $b_m \neq 0$, then we call z_0 a pole of order m.
3. If $b_j \neq 0$ for infinitely many j, then we call z_0 as essential singularity.

Removable Singularities

Theorem (Classification of Removable Singularities)

Suppose that f has an isolated singularity at z_0. Then the following are equivalent.

1. *f has a removable singularity at z_0.*
2. *We can define, or if necessary re-define, f at z_0 so that f is analytic at z_0. (Colloquially, we can "remove" the singularity.)*
3. *The $\lim_{z \to z_0} f(z)$ exists.*
4. *There is a $M > 0$ and a $r > 0$ such that*

$$|f(z)| \leq M \quad \text{for all } z \in B_r'(z_0).$$

And Now Poles get their Turn

Theorem

Suppose that D is domain containing z_0 and that f is analytic in $D \setminus \{z_0\}$ so that f has an isolated singularity at z_0. Then f has a pole of order $m \geq 1$ at z_0 if and only if there is an analytic function g on D such that $g(z_0) \neq 0$ and

$$f(z) = \frac{g(z)}{(z - z_0)^m} \quad \text{for all } z \in D \setminus \{z_0\}.$$

Corollary

If f has a pole of order $m \geq 1$ at z_0, then

$$\lim_{z \to z_0} \left|(z - z_0)^k f(z)\right| = \infty$$

whenever $0 \leq k < m$.

Characterizing Poles

Theorem

Suppose that f has an isolated singularity at z_0. Then f has a pole at z_0 if and only if

$$\lim_{z \to z_0} |f(z)| = \infty.$$

Corollary

Suppose that f has an isolated singularity at z_0. Then f has an essential singularity at z_0 if and only if $|f(z)|$ is not bounded near z_0 and $\lim_{z \to z_0} |f(z)| \neq \infty$.

Dense Sets

Definition

A subset $D \subset \mathbf{C}$ is dense in $\mathbf{C}$ if for all $w \in \mathbf{C}$ and all $\epsilon > 0$ we have $D \cap B_\epsilon(w) \neq \emptyset$.

Remark

D is dense in $\mathbf{C}$ if and only if given $w \in \mathbf{C}$ there is a sequence $(d_n) \subset D$ such that $d_n \to w$.

Example

1. $R = \{ r + is : r, s \in \mathbf{Q} \}$ is dense in $\mathbf{C}$.
2. $\mathbf{C} \setminus R$ is also dense in $\mathbf{C}$.
3. If $F \subset \mathbf{C}$ is a finite set, then $\mathbf{C} \setminus F$ is dense in $\mathbf{C}$.

The Midterm is Cool

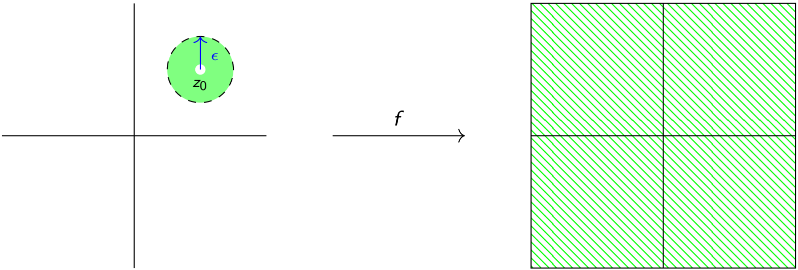
Lemma (Proved on Midterm)

If f is a non-constant entire function, then f has dense range. That is, $f(\mathbf{C})$ is dense in $\mathbf{C}$.

Casorati-Weierstrass

Theorem (Casorati-Weierstrass)

Suppose that f has an essential singularity at z_0. Then for all $\epsilon > 0$, the image, $f\big(B'_\epsilon(z_0)\big)$, of the deleted ϵ-ball about z_0 is dense in $\mathbf{C}$.



Forbidden Friut

I am stating these for the sake of intellectual curiosity. They are not fair game in this course.

Theorem (Picard's Great Theorem)

If f has an essential singularity at z_0, then with possibly one exception, f assume every complex value infinitely often in any deleted neighborhood $B'_\epsilon(z_0)$ of z_0.

Remark

You can verify this for $f(z) = \exp(\frac{1}{z})$ near $z_0 = 0$. Note that the exceptional value here is 0.

Theorem (Picard's Little Theorem)

If the range of an entire function omits more than one point, then it is constant.