Math 43: Spring 2020
Lecture 21 Part I

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Friday May 15, 2020

Today

1. We should be recording!
2. Monday's lecture (#22) will be available in advance as usual.
3. We've worked quite hard to see that analytic functions can be expressed as convergent power series or sometimes convergent Laurent series. The gist of §5.6 is to see what implications this has for analytic functions.
4. The answer will turn out to be "quite a lot actually"!

Zeros

Definition

Suppose that f is analytic at z_0 and that $f(z_0) = 0$. Then we say that z_0 is a zero of order $m \geq 1$ for f if

$$0 = f(z_0) = f'(z_0) = \cdots = f^{(m-1)}(z_0) \quad \text{and} \quad f^{(m)}(z_0) \neq 0.$$

If $f^{(m)}(z_0) = 0$ for all $n \geq 0$, then we say that z_0 is a zero of infinite order.

Example

1. Let $f(z) = \cos(z)$. Then $\frac{\pi}{2}$ is a zero, and since $f'(z) = -\sin(z)$ and $f'(\frac{\pi}{2}) = -1 \neq 0$, $z_0 = \frac{\pi}{2}$ is a zero of order 1 for $f(z) = \cos(z)$. Zeros of order 1 are often called simple zeros.
2. Let $g(z) = 1 - \cos(z)$. Since $g'(z) = \sin(z)$ and $g''(z) = \cos(z)$, it follows that $z_0 = 0$ is a zero of order 2 for g.

Zeros of Infinite Order are Uninteresting

Lemma

Suppose that f has a zero of infinite order at z_0. Then there is a $R > 0$ such that f is identically zero in $B_R(z_0)$.

Proof.

There is a $R > 0$ such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{for all } z \in B_R(z_0).$$

But if z_0 is a zero of infinite order then

$$a_n = \frac{f^{(n)}(z_0)}{n!} = 0 \quad \text{for all } n \geq 0.$$

Hence $f(z) = 0$ for all $z \in B_R(z_0)$. □

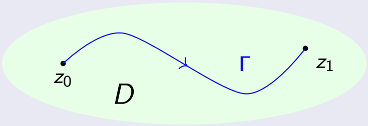
A Nice Theorem not in the Book

Theorem

Suppose that f is analytic in a domain D. If f has a zero of infinite order in D, then f is identically zero in D.

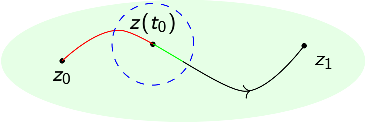
Proof.

Let $z_0 \in D$ be a zero of infinite order for f. Suppose to the contrary that there is a $z_1 \in D$ where $f(z_1) \neq 0$.



Then since D is connected, there is a contour Γ in D from z_0 to z_1. Let $z : [0, 1] \to D$ be an admissible parameterization of Γ. Note that $z(0) = z_0$ and $z(1) = z_1$.

Proof Continued



Let A be the set of all $t \in [0,1]$ such that $z(s)$ is a zero of infinite order for f for all $s \in [0,t]$. We have $0 \in A$ and $1 \notin A$. Let t_0 be the least upper bound of A.

This means for any $t \in [0,t_0)$, $z(t)$ is a zero of infinite order for f, and if $t \in (t_0,1]$ then there is a $t' \in (t_0,t)$ such that $z(t')$ not a zero of infinite order for f. Then by continuity of the derivatives $f^{(n)}(z)$, $t_0 \in A$. But since D is open there is a $r > 0$ such that $B_r(z(t_0)) \subset D$. Then our lemma implies that f is identically zero in $B_r(z(t_0))$. This contradicts our choice of t_0.

Thus there can be no $z_1 \in D$ with $f(z_1) \neq 0$. This completes the proof.

More Good Stuff not in the Book

Theorem

Suppose that f is a non-constant analytic function in a domain D. If $z_0 \in D$ is a zero for f, then z_0 has finite order $m \geq 1$, and there is an analytic function g on D such that $g(z_0) \neq 0$ and such that

$$f(z) = (z - z_0)^m g(z) \quad \text{for all } z \in D. \qquad (\dagger)$$

Return

Proof.

Since f is not constantly equal to zero, z_0 must have finite order $m \geq 1$ by the previous result. Since D is open, there is a $r > 0$ such that $B_r(z_0) \subset D$. Then the Taylor series for f about z_0

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

converges for all $z \in B_r(z_0)$.

Proof Continued

Proof.

Since $f(z_0) = \cdots = f^{m-1}(z_0) = 0$,

$$\begin{aligned} f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n &= \sum_{n=m}^{\infty} a_n (z - z_0)^n \\ &= (z - z_0)^m \sum_{n=m}^{\infty} a_n (z - z_0)^{n-m} \\ &= (z - z_0)^m \sum_{n=0}^{\infty} a_{n+m} (z - z_0)^n = (z - z_0)^m h(z) \end{aligned}$$

where h is analytic in $B_r(z_0)$. Also $h(z_0) = a_m \neq 0$ since $f^{(m)}(z_0) \neq 0$ by assumption.

Proof.

Now we can define

$$g(z) = \begin{cases} \frac{f(z)}{(z-z_0)^m} & \text{if } z \neq z_0, \text{ and} \\ a_m & \text{if } z = z_0. \end{cases}$$

Clearly, g is analytic in $D \setminus \{z_0\}$. But $g(z) = h(z)$ if $z \in B_r(z_0)$. Hence $g'(z_0)$ exists (and equals $h'(z_0)$). Hence g is analytic in D, satisfies our ▸ requirement in (†) and $g(z_0) = h(z_0) = a_m \neq 0$. □

Punch Line

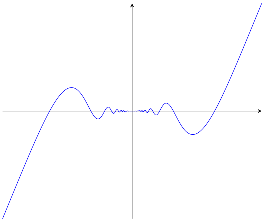
Corollary

If f is a non-constant analytic function on a domain D, then the zeros of f are isolated. That is, if $f(z_0) = 0$ for some $z_0 \in D$, then there is a $\delta > 0$ such that $f(z) \neq 0$ if $z \in B'_\delta(z_0) = \{\, z : 0 < |z - z_0| < \delta \,\}$.

Proof.

By the previous result, $f(z) = (z - z_0)^m g(z)$ with $m \geq 1$, g analytic in D, and $g(z_0) \neq 0$. Since g is continuous at z_0, there is a $\delta > 0$ such that $g(z) \neq 0$ if $z \in B_\delta(z_0)$. Hence $f(z) \neq 0$ if $z \in B'_\delta(z_0)$. $\square$

The Real World is a Frightening Place



Example

Let $f : [-1, 1] \to \mathbf{R}$ be given by

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \text{ and} \\ 0 & \text{if } x = 0. \end{cases}$$

Then f is differentiable and 0 is a zero of f, but so are $\{\, \frac{\pi}{n} : n \in \mathbf{Z} \,\}$. Thus 0 is not an isolated zero.

Enough for Now

Remark

After the break, we see some surprising consequences of the fact that the zeros of non-constant analytic functions must be isolated.

Time for a Break.