Math 43: Spring 2020
Lecture 21 Part II

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Reminder

1. Still recording?

Zeros Again

Remark

In the first part of this lecture, we saw that the zeros of non-constant analytic functions are isolated. This has some cool consequences.

Example

Suppose that f is an entire function such that $f(x) = e^x$ for all $x \in \mathbf{R}$. Then $f(z) = e^z$ for all $z \in \mathbf{C}$.

Solution.

Let $h(z) = f(z) - e^z$. Then h is entire. But h has zeros at *every* $z \in \mathbf{R}$. Hence, for example, $z = 0$ is not an isolated zero of h. Therefore h must be identically zero. □

Even Functions

Example

Suppose that f is an entire function and that

$$f\left(\frac{1}{n}\right) = f\left(-\frac{1}{n}\right) \quad \text{for all } n \in \mathbf{N}.$$

Then $f(z) = f(-z)$ for all $z \in \mathbf{C}$.

Solution.

Let $h(z) = f(z) - f(-z)$. Then h is entire. By assumption $h\left(\frac{1}{n}\right) = 0$ for all $n \in \mathbf{N}$. Since $\frac{1}{n} \to 0$, we must have $h(0) = 0$ by continuity. But then 0 is not an isolated zero of h. Therefore h is identically zero and $f(z) = f(-z)$ as claimed. $\square$

Example

Show that $\cos(2z) = \cos^2(z) - \sin^2(z)$ for all $z \in \mathbf{C}$.

Solution.

The function $f(z) = \cos(2z) - \cos^2(z) + \sin^2(z)$ is entire and vanishes for all $z \in \mathbf{R}$. $\square$

Another

Example

Show that $\cos(z+w) = \cos(z)\cos(w) - \sin(z)\sin(w)$ for all $z, w \in \mathbf{C}$.

Solution.

If $y \in \mathbf{R}$, then $f(z) = \cos(z)\cos(y) - \sin(z)\sin(y) - \cos(z+y)$ is entire and vanishes for all $z \in \mathbf{R}$. Hence $f(z) = 0$ for all $z \in \mathbf{C}$. Now fix $z \in \mathbf{C}$ and let
$g(w) = \cos(z)\cos(w) - \sin(z)\sin(w) - \cos(z+w)$. Then g is enter and by the above vanishes for all $w \in \mathbf{R}$. Hence g is identically zero. Since z was arbitrary, we are done. $\square$

And Now Singularities

Definition

If f is analytic in $B'_R(z_0)$ for some $R > 0$, then we say that f has an isolated singularity at z_0

Example (Isolated Singularities at $z_0 = 0$)

1. $f(z) = \dfrac{1 - \cos(z)}{z^2}$.
2. $g(z) = \dfrac{\sin(z)}{z^3}$.
3. $h(z) = z^2 \cos\left(\dfrac{1}{z}\right)$

Example (Not Isolated)

The function $q(z) = \mathrm{Log}(z)$ has singularities at every $z_0 \in (-\infty, 0]$. None of these is isolated.

Important

Remark

If f has an isolated singularity at z_0, then for some $R > 0$, f is analytic in the degenerate annulus $A = \{\, z : 0 < |z - z_0| < R \,\}$. Hence f has a Laurent series in A about z_0:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{j=1}^{\infty} \frac{b_j}{(z - z_0)^j} \qquad (\ddagger)$$

for all $z \in A$. We know that the coefficients a_n and b_j appearing in $(\ddagger)$ depend only on f and z_0. Hence we call $(\ddagger)$ the Laurent series for f at z_0. Laurent series for analytic functions at isolated singularities are our primary reason for studying Laurent series.

Return

Definition

Suppose that f has an isolated singularity at z_0 with Laurent series (1) on the previous slide.

1. If $b_j = 0$ for all $j \geq 1$, then we call z_0 a removable singularity.
2. If $b_j = 0$ for all $j > m$ and $b_m \neq 0$, then we call z_0 a pole of order m.
3. If $b_j \neq 0$ for infinitely many j, then we call z_0 as essential singularity.

Removable

Example

Let $f(z) = \dfrac{1-\cos(z)}{z^2}$. Then

$$\begin{aligned} f(z) &= \frac{1}{z^2}\Big(\frac{z^2}{2} - \frac{z^4}{4!} - \cdots\Big) \\ &= \frac{1}{2} - \frac{z^2}{4!} + \frac{z^4}{6!} - \cdots \\ &= \sum_{n=0}^{\infty} \frac{z^{2n}}{(2(n+1))!}. \end{aligned}$$

Thus f has a removable singularity at $z_0 = 0$.

Pole

Example

Let $g(z) = \dfrac{\sin(z)}{z^3}$. Then

$$\begin{aligned} g(z) &= \frac{1}{z^3}\Big(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\Big) \\ &= \frac{1}{z^2} - \frac{1}{6} + \frac{z^2}{5!} - \cdots \\ &= \frac{1}{z^2} + \sum_{n=0}^{\infty} (-1)^{n-1} \frac{z^{2n}}{(2n+3)!}. \end{aligned}$$

Hence g has a pole of order 2 at $z_0 = 0$.

Essential

Example

Let $h(z) = z^2 \cos\left(\frac{1}{z}\right)$. Then

$$\begin{aligned} h(z) &= z^2 \left(1 - \frac{1}{2 \cdot z^2} + \frac{1}{4! \cdot z^4} - \cdots \right) \\ &= z^2 - \frac{1}{2} + \frac{1}{4! \cdot z^2} - \frac{1}{6! z^4} + \cdots \\ &= z^2 - \frac{1}{2} + \sum_{j=1}^{\infty} (-1)^{j+1} \frac{1}{(2n+2)! \cdot z^{2j}}. \end{aligned} \tag{1}$$

Hence h has an essential singularity at $z_0 = 0$.

Enough for Now

Remark

While we have neatly characterized all isolated singularities into three distinct flavors, there is no particular reason to expect much from this classification except that it is natural with respect to the corresponding Laurent series. On Monday, we will see that these types of singularities can also be distinguished qualitatively. That will make our classification much more useful.

That is enough for today. We need to stop the recording soon.