Math 43: Spring 2020
Lecture 22 Part I

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Classification of Singularities

Definition (From Friday's Lecture)

Suppose that f has an isolated singularity at z_0 with Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{j=1}^{\infty} \frac{b_j}{(z - z_0)^j}.$$

1. If $b_j = 0$ for all $j \geq 1$, then we call z_0 a removable singularity.
2. If $b_j = 0$ for all $j > m$ and $b_m \neq 0$, then we call z_0 a pole of order m.
3. If $b_j \neq 0$ for infinitely many j, then we call z_0 as essential singularity.

Real Frightening

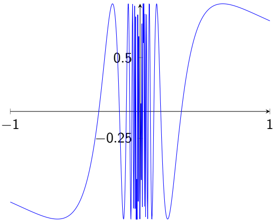


Figure: The Graph of $y = \sin\left(\frac{1}{x}\right)$

The function $f : [-1, 1] \to \mathbf{R}$ given by

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \text{ and} \\ 0 & \text{if } x = 0 \end{cases}$$

is bounded—we have $|f(x)| \leq 1$ for all x—but of course f has no limit at $x = 0$ and hence can't be adjusted to be continuous there let alone differentiable.

Removable Singularities

Theorem (Classification of Removable Singularities)

Suppose that f has an isolated singularity at z_0. Then the following are equivalent.

1. *f has a removable singularity at z_0.*
2. *We can define, or if necessary re-define, f at z_0 so that f is analytic at z_0. (Colloquially, we can "remove" the singularity.)*
3. *The $\lim_{z \to z_0} f(z)$ exists.*
4. *There is a $M > 0$ and a $r > 0$ such that*

$$|f(z)| \leq M \quad \text{for all } z \in B'_r(z_0).$$

Proof.

Since z_0 is an isolated singularity, by definition we can assume that there is a $R > 0$ such that f is analytic in $B'_R(z_0)$. We can complete the proof by showing that
(1)$\Longrightarrow$(2)$\Longrightarrow$(3)$\Longrightarrow$(4)$\Longrightarrow$(1).
(1)$\Longrightarrow$(2): But assumption, the Laurent series for f has the form

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n \quad \text{for all } z \in B'_R(z_0).$$

But then

$$g(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

is analytic in all of $B_R(z_0)$. Since $f(z) = g(z)$ if $z \neq z_0$, we just have to let $f(z_0) = g(z_0) = a_0$.

Proof Continued

Proof.

(2)$\Longrightarrow$(3): If f is analytic at z_0, then f is continuous at z_0. Hence

$$\lim_{z \to z_0} f(z) = f(z_0)$$

must exist.

(3)$\Longrightarrow$(4): Suppose that $\lim_{z \to z_0} f(z) = L$. Let r be such that $0 < |z - z_0| < r$ implies that $|f(z) - L| < 1$. Then if $z \in B'_r(z_0)$, we have

$$|f(z)| = |f(z) - L + L| \le |f(z) - L| + |L| \le 1 + |L|.$$

Hence we can let $M = |L| + 1$.

The Fun Bit

Proof.

(4)$\Longrightarrow$(1): Let $0 < r < R$, and let C_r be the positively oriented circle of radius r centered at z_0. By Laurent's Theorem, and since we can use any positively oriented simple closed contour in $B_R'(z_0)$ containing z_0 in its interior to compute the b_j, for any $j \geq 1$ we have

$$b_j = \frac{1}{2\pi i} \int_{C_r} f(w)(w - z_0)^{j-1}\, dw.$$

But then

$$|b_j| \leq \frac{1}{2\pi} M r^{j-1}(2\pi r) = M r^j.$$

Since this holds for any $0 < r < R$, we must have $b_j = 0$. Since j was arbitrary, we are done. □

Example

Example

Suppose that f is analytic in D, that $0 \in D$, and that $f(0) = 0$. Then show that

$$F(z) = \begin{cases} \frac{f(z)}{z} & \text{if } z \neq 0, \text{ and} \\ f'(0) & \text{if } z = 0 \end{cases}$$

is analytic in D.

Solution.

Clearly, F has an isolated singularity at 0. Since $\lim_{z \to 0} F(z) = \lim_{z \to 0} \frac{f(z)-f(0)}{z-0} = f'(0)$ exists, 0 is a removable singularity. Since $f'(0)$ is the only value that makes F continuous at 0, it the only possible value that makes F analytic at 0. $\square$

Fun Example

Example

Suppose that f and g are entire and $|f(z)| \leq |g(z)|$ for all z. Show that there is a $c \in \mathbf{C}$ such that $f(z) = cg(z)$.

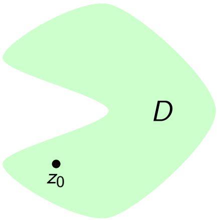
Solution.

If g is identically 0, then so is f and the assertion is clearly true. Otherwise, let

$$h(z) = \frac{f(z)}{g(z)}.$$

Since g is not constantly equal to 0, its zeros are isolated. Hence h has only isolated singularities. Since $|h(z)| \leq 1$ for all z in its domain, all the singularities of h are removable. Hence we can define h at each of its singularities to make it entire. Since it is entire and bounded, h is a constant. That is, $h(z) = c$ for all z. The result follows. $\square$

And Now Poles get their Turn



Theorem

Suppose that D is domain containing z_0 and that f is analytic in $D \setminus \{z_0\}$ so that f has an isolated singularity at z_0. Then f has a pole of order $m \geq 1$ at z_0 if and only if there is an analytic function g on D such that $g(z_0) \neq 0$ and

$$f(z) = \frac{g(z)}{(z - z_0)^m} \quad \textit{for all } z \in D \setminus \{z_0\}.$$

The Proof

Proof.

Suppose that f has a pole of order $m \geq 1$ at $z_0 \in D$. Then there is a $R > 0$ such that $B_R(z_0) \subset D$ and such that for all $z \in B'_R(z_0)$ we have

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \frac{b_1}{z - z_0} + \cdots + \frac{b_m}{(z - z_0)^m}$$

with $b_m \neq 0$. Hence

$$f(z) = \frac{1}{(z - z_0)^m} \Big[b_m + b_{m-1}(z - z_0) + \cdots + b_1(z - z_0)^{m-1} + \sum_{n=0}^{\infty} a_n(z - z_0)^{n+m} \Big]$$

Proof Continued

Proof.

Hence $f(z) = \dfrac{1}{(z-z_0)^m} h(z)$ where $h(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n$ for all $z \in B_R(z_0)$ and

$$c_n = \begin{cases} b_{m-n} & \text{if } n \le m-1, \text{ and} \\ a_{n-m} & \text{if } n \ge m. \end{cases}$$

Then h is analytic in $B_R(z_0)$ and $h(z_0) = b_m \ne 0$. Now let

$$g(z) = \begin{cases} f(z)(z-z_0)^m & \text{if } z \ne z_0, \text{ and} \\ b_m & \text{if } z = z_0. \end{cases}$$

Then g is analytic in $D \setminus \{z_0\}$ and $g(z) = h(z)$ for all $z \in B_R(z_0)$. Hence g is analytic in all of D and $f(z) = g(z)/(z-z_0)^m$ if $z \ne z_0$ as required.

The other direction is easier and is left as a homework problem. □

Poles Blow Up

Corollary

If f has a pole of order $m \geq 1$ at z_0, then

$$\lim_{z \to z_0} \left| (z - z_0)^k f(z) \right| = \infty \qquad (\ddagger)$$

whenever $0 \leq k < m$.

Proof.

We can assume $f(z) = \frac{g(z)}{(z-z_0)^m}$ with g analytic and $g(z_0) \neq 0$. Then $(\ddagger)$ becomes

$$\lim_{z \to z_0} \frac{|g(z)|}{|z - z_0|^{m-k}}$$

which, since $|g(z_0)| > 0$, diverges to infinity if $m - k \geq 1$. □

Remark

After the break, we'll see that the previous corollary has a converse!

Time for a Break.