Math 43: Spring 2020
Lecture 23 Summary

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Today

1. Are we recording?
2. Monday is a "holiday" and there will be no class meeting, lecture, or additional assignments.
3. This brief respite is an opportunity to catch up on all the deep results from Chapter 5.
4. The beginning of Chapter 6 is computational. The material is more routine, but the algebra can be challenging.

Residues

Definition

If f has an isolated singularity at z_0 with Laurent series given by

$$f(z) = \sum_{n=0}^{a_n} a_n(z - z_0)^n + \sum_{j=1}^{\infty} \frac{b_j}{(z - z_0)^j}, \qquad (\dagger)$$

then we call b_1 the residue of f at z_0, and we write

$$b_1 := \operatorname{Res}(f; z_0).$$

Remark

If $(\dagger)$ is valid in $B'_R(z_0)$ and if C is any positively oriented contour in $B'_R(z_0)$ with z_0 in its interior, then

$$\int_C f(z)\,dz = 2\pi i \cdot \operatorname{Res}(f; z_0).$$

Low Hanging Fruit

Remark

If f has a removable singularity at z_0, then $\operatorname{Res}(f; z_0) = 0$.

Lemma

If f has a simple pole at z_0, then

$$\operatorname{Res}(f; z_0) = \lim_{z \to z_0} (z - z_0) f(z).$$

Conversely, if f has an isolated singularity at z_0 and

$$\lim_{z \to z_0} (z - z_0) f(z) = L \neq 0,$$

then f has a simple pole at z_0 and $L = \operatorname{Res}(f; z_0)$.

The Simple Pole Lemma

Theorem (The Simple Pole Lemma)

Suppose that h and g are analytic at z_0. Suppose also that h has a simple zero at z_0 while $g(z_0) \neq 0$. Then

$$f(z) = \frac{g(z)}{h(z)}$$

has a simple pole at z_0 and

$$\operatorname{Res}(f; z_0) = \frac{g(z_0)}{h'(z_0)}.$$

Non Simple Poles

Example

If f has a pole of order 2 at z_0, then

$$\operatorname{Res}(f; z_0) = \lim_{z \to z_0} \frac{d}{dz}\left[(z - z_0)^2 f(z)\right]$$

The General Case

Just as with computing partial fraction decompositions, the authors of our text provide us with a handy—and easy to mess-up—general formula. While I feel honor bound to report its existence, it is generally safer to work it out if and when you need it.

Lemma

If f has a pole of order $m \geq 1$ at z_0, then

$$\operatorname{Res}(f; z_0) = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \Big[(z - z_0)^m f(z) \Big].$$

The Cauchy Residue Theorem

Theorem (Cauchy Residue Theorem)

Suppose that f is analytic on and inside a positively oriented simple closed contour Γ except for isolated singularities $z_1, \dots, z_n$ inside of Γ. Then

$$\int_\Gamma f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f; z_k).$$

The Adventure Continues

Remark

The Cauchy Residue Theorem makes evaluating contour integrals over simple closed contours tractable. But at the moment, this is more of an intellectual curiosity than anything else. However in §§6.2–4, we will use contour integrals to solve some interesting problems from calculus of functions of a real variable! Problems in the real world if you will. We will start that adventure with Friday's lecture.