Math 43: Spring 2020
Lecture 23 Part I

Dana P. Williams

Dartmouth College

Wednesday May 20, 2020

Residues

If f has an isolated singularity at z_0, then we know that f has a Laurent series

$$f(z) = \sum_{n=0}^{a_n} a_n(z - z_0)^n + \sum_{j=1}^{\infty} \frac{b_j}{(z - z_0)^j} \qquad (\dagger)$$

which is valid is some deleted neighborhood $B_R'(z_0)$. While we have handy formulas for all the coefficients, b_1 is special. If Γ is any simple closed contour in $B_R'(z_0)$ with z_0 in its interior, then

$$\int_\Gamma f(z)\, dz = 2\pi i \cdot b_1.$$

Definition

If f has an isolated singularity at z_0 with Laurent series given by ($\dagger$), then we call b_1 the residue of f at z_0, and we write

$$b_1 := \mathrm{Res}(f; z_0).$$

Example

Example

Compute $I = \int_{|z|=1} z^2 \exp\left(\frac{1}{z}\right) dz$. (Note: writing $e^{\frac{1}{z}}$ just doesn't cut it on slides; hence I may over use exp a bit.)

Solution.

As usual in this situation, we assume $|z| = 1$ is positively oriented unless otherwise told. Now

$$z^2 \exp(\frac{1}{z}) = z^2\Big(1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \cdots\Big)$$
$$= z^2 + z + \frac{1}{2} + \frac{1}{6z} + \cdots.$$

Thus $I = 2\pi i \cdot \text{Res}(z^2 \exp(\frac{1}{z}); 0) = 2\pi i \cdot \frac{1}{6} = \frac{\pi i}{3}$. $\qquad\square$

Remark

If f has a removable singularity at z_0, then $\text{Res}(f; z_0) = 0$.

Remark (Simple Poles)

Suppose that f has a simple pole at z_0—a simple pole is a pole of order 1. Then near z_0,

$$f(z) = \frac{b_1}{z - z_0} + a_0 + a_2(z - z_0) + \cdots$$
$$= \frac{b_1}{z - z_0} + g(z)$$

where g is analytic at z_0. In particular,

$$(z - z_0)f(z) = b_1 + (z - z_0)g(z) \qquad (\ddagger)$$

in a deleted neighborhood of z_0. ▸ Return

Lemma

If f has a simple pole at z_0, then

$$\operatorname{Res}(f; z_0) = \lim_{z \to z_0} (z - z_0) f(z).$$

Conversely, if f has an isolated singularity at z_0 and

$$\lim_{z \to z_0} (z - z_0) f(z) = L \neq 0,$$

then f has a simple pole at z_0 and $L = \operatorname{Res}(f; z_0)$.

Proof.

The first statement follows immediately from ($\ddagger$)
on the ▸ previous page .

The Converse

Proof.

For the converse, assume $\lim_{z \to z_0}(z - z_0)f(z) = L \neq 0$. Then

$$\lim_{z \to z_0} |f(z)| = \lim_{z \to z_0} \frac{|(z - z_0)f(z)|}{|z - z_0|} = \infty$$

since the numerator tends to $|L| \neq 0$. Thus f has a pole at z_0 of order $m \geq 1$. But if $m \geq 2$, then $\lim_{z \to z_0} |(z - z_0)f(z)| = \infty$!. Since this limit is finite, z_0 is a simple pole. $\quad\square$

Example

Consider $f(z) = \dfrac{e^z}{z(z+2)}$.

Since $g(z) = e^z/(z+2)$ is analtyic and nonzero at 0, and since $f(z) = g(z)/z$, it follows that f has a simple pole at 0. Hence

$$\text{Res}(f; 0) = \lim_{z \to 0}(z - 0)f(z) = \lim_{z \to 0} \frac{e^z}{z+2} = \frac{1}{2}.$$

Of course, since the above limit is nonzero, that also implies that 0 is a simple pole for f. Similarly,

$$\text{Res}(f; -2) = \lim_{z \to -2}(z + 2)f(z) = \lim_{z \to -2} \frac{e^z}{z} = -\frac{1}{2} \cdot e^{-2}.$$

Break Time

Remark

After the break,

Time for a Break.