Math 43: Spring 2020
Lecture 23 Part II

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The Simple Pole Lemma

Our next result makes it easy to compute residues in certain cases. You should expect to use it regularly!

Theorem (The Simple Pole Lemma)

Suppose that h and g are analytic at z_0. Suppose also that h has a simple zero at z_0 while $g(z_0) \neq 0$. Then

$$f(z) = \frac{g(z)}{h(z)}$$

has a simple pole at z_0 and

$$\operatorname{Res}(f; z_0) = \frac{g(z_0)}{h'(z_0)}. \tag{$\dagger$}$$

The Proof

Proof.

Since h has a simple zero at z_0, we have $h'(z_0) \neq 0$. Hence at least the right-hand side of ($\dagger$) is well-defined. Moreover,

$$\begin{aligned} \lim_{z \to z_0} (z - z_0) f(z) &= \lim_{z \to z_0} \frac{(z - z_0) g(z)}{h(z)} \\ &= \lim_{z \to z_0} \frac{g(z)}{\frac{h(z) - h(z_0)}{z - z_0}} \\ &= \frac{g(z_0)}{h'(z_0)} \end{aligned}$$

since g is continuous at z_0 and $h'(z_0) \neq 0$. $\square$

Example

Example

Let $f(z) = \dfrac{z^2}{z^4+1}$. Let $w = e^{i\frac{\pi}{4}} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$. Find $\operatorname{Res}(f; w)$.

Solution.

Notice that the Simple Pole Lemma applies! Hence

$$\begin{aligned}\operatorname{Res}(f;z) &= \frac{z^2}{4z^3}\Big|_{z=w} \\ &= \frac{1}{4w} = \frac{\overline{w}}{4} \\ &= \frac{1}{4}\Big(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\Big).\end{aligned}$$

$\square$

Non Simple Poles

Example

Let $f(z) = \dfrac{e^z}{(z^2+1)^2}$. Compute $\operatorname{Res}(f;i)$.

Notice that

$$f(z) = \frac{\frac{e^z}{(z+i)^2}}{(z-i)^2}.$$

Hence i is a pole of order 2 for f! This is because $g(z) = e^z/(z+i)^2$ is analytic and nonzero at i! But how can we computer the residue at poles of higher order when the Laurent series is hard (or even impossible) to compute?

Back to our Old Tricks

Let's look at the general case where f has a pole of order 2 at z_0. Then

$$f(z) = \frac{b_2}{(z-z_0)^2} + \frac{b_1}{z-z_0} + g(z)$$

where g is analytic at z_0. Then

$$(z-z_0)^2 f(z) = b_2 + b_1(z-z_0) + (z-z_0)^2 g(z).$$

Therefore

$$\frac{d}{dz}\Big[(z-z_0)^2 f(z)\Big] = b_1 + 2(z-z_0)g(z) + (z-z_0)^2 g'(z).$$

Now we see that

$$\operatorname{Res}(f; z_0) = b_1 = \lim_{z\to z_0} \frac{d}{dz}\Big[(z-z_0)^2 f(z)\Big].$$

Back to our Example

Recall that we started by asking for $\operatorname{Res}(f;i)$ where $f(z) = \dfrac{e^z}{(z^2+1)^2}$. Based on the previous slide,

$$\begin{aligned}
\operatorname{Res}(f;i) &= \lim_{z\to i} \frac{d}{dz}\Big[(z-i)^2 \frac{e^z}{(z^2+1)^2}\Big] = \lim_{z\to i} \frac{d}{dz}\Big[\frac{e^z}{(z+i)^2}\Big] \\
&= \lim_{z\to i} \frac{e^z(z+i)^2 - 2(z+i)e^z}{(z+i)^4} \\
&= \lim_{z\to i} \frac{e^z(z+i) - 2e^z}{(z+i)^3} \\
&= \frac{e^i(2i-2)}{-8i} = \frac{e^i(i-1)}{-4i} = e^i = -e^i\frac{1+i}{4}.
\end{aligned}$$

The General Case

Just as with computing partial fraction decompositions, the authors of our text provide us with a handy—and easy to mess-up—general formula. While I feel honor bound to report its existence, it is generally safer to work it out if and when you need it.

Lemma

If f has a pole of order $m \geq 1$ at z_0, then

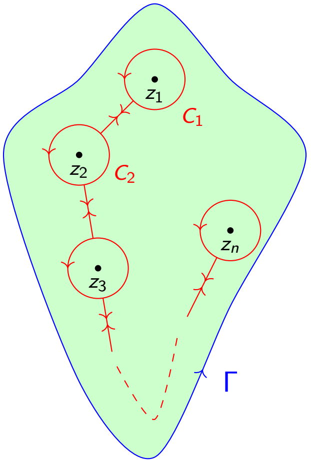
$$\operatorname{Res}(f; z_0) = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \Big[(z - z_0)^m f(z) \Big].$$

Theorem (Cauchy Residue Theorem)

Suppose that f is analytic on and inside a positively oriented simple closed contour Γ except for isolated singularities $z_1, \dots, z_n$ inside of Γ. Then

$$\int_\Gamma f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f; z_k).$$

Proof



Sketch of the Proof.

Let D be the interior of Γ. We assume that we can continuously deform Γ in $D \setminus \{ z_1, \dots, z_n \}$ to the union of n postively oriented circles C_k centered at the singularities z_k together with canceling line segments. Then by the Deformation Invariance Theorem,

$$\int_\Gamma f(z)\,dz = \sum_{k=1}^{n} \int_{C_k} f(z)\,dz = \sum_{k=1}^{n} 2\pi i \operatorname{Res}(f; z_k). \quad \square$$

Example

Example

Evaluate $I = \displaystyle\int_{|z|=2} \frac{e^z}{z^2+1}\,dz$.

Solution.

As always, without any indication to the contrary, we are supposed to assume that $|z| = 2$ is positively oriented. Then by the Cauchy Residue Theorem,

$$I = 2\pi i\big(\operatorname{Res}(f;i) + \operatorname{Res}(f;-i)\big) = 2\pi i\big(\operatorname{Res}(i) + \operatorname{Res}(-i)\big).$$

By the Simple Pole Lemma, $\operatorname{Res}(i) = \frac{e^i}{2i}$ and $\operatorname{Res}(-i) = \frac{e^{-i}}{-2i}$. Hence

$$I = 2\pi i\Big(\frac{e^i - e^{-i}}{2i}\Big) = 2\pi i \cdot \sin(1). \quad \square$$

Break Time

Remark

Well, if you like computing contour integrals, that last computation was pretty neat! Now I have to convince you that there is a good reason to compute a contour integral—other than doing well on Math 43 exams.

But we'll deal with that in the coming week or so. Now we should stand down for today.