Math 43: Spring 2020
Lecture 24 Summary

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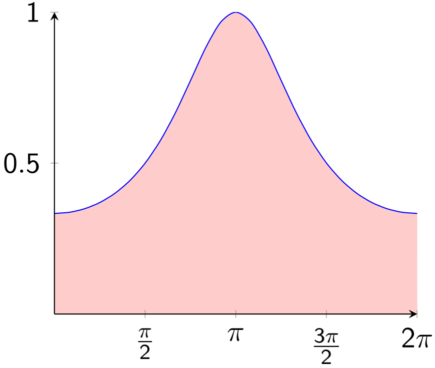
Today

- I hope we are recording.
- Recall that there is no class meeting on Monday.
- Wednesday's lecture will be posted soon but perhaps not today.
- Use this "pause in the action" wisely.

Time for a Calculus Problem in the "Real" World

Example

We showed that $I = \int_0^{2\pi} \frac{1}{2+\cos(\theta)}\, d\theta = \frac{2\pi}{\sqrt{3}}$.



The General Method

1. We use $\displaystyle\int_{|z|=1} F(z)\,dz = \int_0^{2\pi} F(e^{i\theta}) ie^{i\theta}\,d\theta.$
2. Then if $U(x,y)$ is a suitable rational function, we have
$$\int_0^{2\pi} U\big(\cos(\theta),\sin(\theta)\big)\,d\theta = \int_{|z|=1} F(z)\,dz,$$
3. where
$$F(z) = U\Big[\frac{1}{2}\Big(z+\frac{1}{z}\Big), \frac{1}{2i}\Big(z-\frac{1}{z}\Big)\Big]\cdot\frac{1}{iz}.$$
4. In the first example, $U(x,y) = \dfrac{1}{2+x}$. In the example $\displaystyle\int_0^{2\pi} \frac{1}{1+\sin^2(\theta)}\,d\theta = \pi\sqrt{2}$, we used $U(x,y) = \dfrac{1}{1+y^2}$.

Reality Check!

Remark (Real Answers)

One of the cool things about this process is that we use complex theory to provide answers to questions about real calculus! In particular, our answers must be real numbers! This provides a handy reality check (pun intended!) to guard against algebraic mistakes.

Example

We also saw that

$$I = \int_0^\infty \frac{1}{x^4+1}\,dx := \lim_{R\to\infty} \int_0^R \frac{1}{x^4+1}\,dx = \frac{\pi}{2\sqrt{2}}.$$

We used the following method.

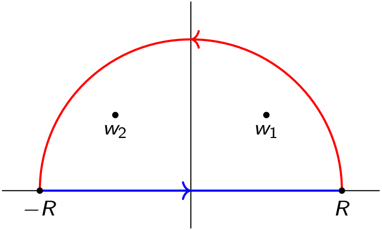
- First, we let $f(z) = 1/(z^4+1)$ and used symmetry to note that

$$I = \frac{1}{2}\lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx = \frac{1}{2}\lim_{R\to 0}\int_{[-R,R]} f(z)\,dz.$$

- This only makes sense if f has no singularities on the real axis.

The Contour Γ_R

- Then we introduce the contour $\Gamma_R = [-R, R] + C_R^+$ where C_R^+ is the top half of the positively oriented circle $|z| = R$ from R to $-R$. We can assume that R is large enough that Γ_R contains all the singularities of f in the upper half-plane.



- Then

$$\int_{[-R,R]} f(z)\,dz = \int_{\Gamma_R} f(z)\,dz - \int_{C_R^+} f(z)\,dz.$$

Key Observations

- Using the usual estimates for the modulus of a contour integral,

$$\lim_{R\to\infty}\int_{C_R^+} f(z)\,dz = 0.$$

- On the other hand, once R is sufficiently large,

$$\int_{\Gamma_R} f(z)\,dz$$

is constant, and by the Cauchy Residue Theorem

$$\int_{\Gamma_R} f(z)\,dz = 2\pi i \cdot \sum \Big(\text{Residues inside } \Gamma_R\Big)$$

which, if R is large enough, can be abbreviated by

$$= 2\pi i \sum_{\operatorname{Im}(z)>0} \operatorname{Res}(f; z).$$

Taking the Limit

- Now we can put it all together and take the limit as follows

$$\begin{aligned} I &= \frac{1}{2}\lim_{R\to\infty}\int_{[-R,R]} f(z)\,dz \\ &= \frac{1}{2}\int_{\Gamma_R} f(z)\,dz - \frac{1}{2}\lim_{R\to\infty}\int_{C_R^+} f(z)\,dz \\ &= \pi i \cdot \Big(\sum_{\mathrm{Im}(z)>0} \mathrm{Res}(f;z)\Big) + 0. \end{aligned}$$

- In our example, f had just two singularities in the upper half-plane and we easily computed the residues using the Simple Pole Lemma.
- This method extends easily to a variety of types of improper real integrals. We'll investigate some of these Wednesday and into Friday next week. A tiny review of improper integrals would not go amiss.