Math 43: Spring 2020
Lecture 24 Part II

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Improper Integrals

Example

Compute $I = \int_0^\infty \frac{1}{x^4+1}\,dx := \lim_{R\to\infty} \int_0^R \frac{1}{x^4+1}\,dx.$

Remark

Again, we technically have the tools to evaluate this. We could form a partial fraction decomposition

$$\frac{1}{x^4+1} = \frac{x-\sqrt{2}}{2\sqrt{2}\left(-x^2+\sqrt{2}x-1\right)} + \frac{x+\sqrt{2}}{2\sqrt{2}\left(x^2+\sqrt{2}x+1\right)}$$

and waste away hours with trig substitutions to compute an antiderivative

$$\frac{1}{4\sqrt{2}}\Big[-\log\left(x^2-\sqrt{2}x+1\right)+\log\left(x^2+\sqrt{2}x+1\right) \\ -2\tan^{-1}\left(1-\sqrt{2}x\right)+2\tan^{-1}\left(\sqrt{2}x+1\right)\Big]$$

and then take the limit. Yuck!

A Complex Method

In order to form a useful contour integral, we need to observe that by symmetry we also have

$$I = \lim_{R\to\infty} \int_{-R}^{0} \frac{1}{x^4+1}\,dx$$

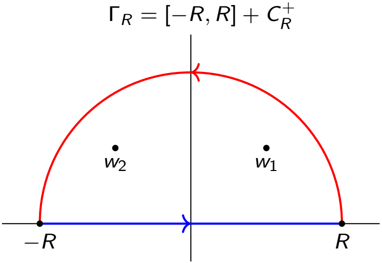
Therefore

$$I = \frac{1}{2}\lim_{R\to\infty} \int_{-R}^{R} \frac{1}{x^4+1}\,dx.$$

We can even turn this into a contour integral problem by introducing the function $f(z) = \frac{1}{z^4+1}$. Then

$$I = \frac{1}{2}\lim_{R\to\infty} \int_{[-R,R]} f(z)\,dz.$$

But this doesn't help as it is just the same problem in disguise.



Now we introduce the contour $\Gamma_R = [-R, R] + C_R^+$ where C_R^+ is the top half of the positively oriented circle $|z| = R$ from R to $-R$. We assume that we have taken $R > 1$ so that Γ_R contains both 4^{th}-roots of -1 in the upper half-plane: $w_1 = \exp(i\frac{\pi}{4})$ and $w_2 = w_1^3 = \exp(i\frac{3\pi}{4})$ in its interior.

Then

$$\int_{[-R,R]} f(z)\,dz = \int_{\Gamma_R} f(z)\,dz - \int_{C_R^+} f(z)\,dz.$$

Return

Getting Closer

But if $R > 1$, then

$$\left|\int_{C_R^+} f(z)\,dz\right| \leq \frac{1}{R^4 - 1} \cdot \pi R = \frac{\pi R}{R^4 - 1}.$$

Thererfore

$$\lim_{R\to\infty} \int_{C_R^+} f(z)\,dz = 0.$$

On the other hand, if $R > 1$, then

$$\int_{\Gamma_R} f(z)\,dz = 2\pi i \cdot \big(\mathrm{Res}(w_1) + \mathrm{Res}(w_1^3)\big)$$

where $w_1 = e^{i\frac{\pi}{4}} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$.

Taking the Limit

But using previous slide,

$$I = \frac{1}{2}\lim_{R\to\infty}\int_{[-R,R]} f(z)\,dz = \pi i \cdot \left[\operatorname{Res}(w_1) + \operatorname{Res}(w_1^3)\right] + 0.$$

Now using the Simple Pole Lemma we have

$$\operatorname{Res}(w_1) = \frac{1}{4z^3}\Big|_{z=w_1} = \frac{1}{4w_1^3} = -\frac{w_1}{4},$$

while

$$\operatorname{Res}(w_1^3) = \frac{1}{4w_1^9} = \frac{1}{4w_1} = \frac{\overline{w_1}}{4}.$$

Therefore

$$I = \pi i \cdot \left(\frac{-w_1 + \overline{w_1}}{4}\right) = \pi i \frac{1}{4} \cdot -i\frac{2}{\sqrt{2}} = \frac{\pi}{2\sqrt{2}}.$$

Time to Rest

Remark

There is no class on Monday! I will hold office hours as usual on Tuesday. On Wednesday, we will see how to generalize this method. We will diverge a bit from the treatment in the text.

Enough for today.